Optimal Securitization with Heterogeneous Investors*

by

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Abstract

We solve the problem of optimal securitization for an issuer facing heterogeneous investors with arbitrary time and risk preferences. We show that the optimal securitization is characterized by multiple nonlinear tranches, and each investor gets a portfolio of these tranches. In particular, when all agents have CARA utilities, the linear tranching is optimal, with the number of tranches being less than or equal to the number of potential investors. To the best of our knowledge, this is the first model in the literature that explains the appearance of multiple tranches in the security design and the relation of the tranche thresholds to microeconomic characteristics.

We show that the boundaries of the tranches can be efficiently calculated through a fixed point of a contraction mapping. We use these contraction mapping techniques to derive a number of comparative static results for optimal securitization. The model generates theoretical predictions and numerical simulations that agree with several recent empirical findings concerning the CDO structure.

Keywords: securitization, tranching, heterogeneity, mortgage backed securities, asset backed securities

JEL Classifications: G32, G21, G24

1 Introduction

Since its inception almost 40 years ago, the asset securitization industry has grown to a multitrillion dollar business. The practice of creating multiple tranches from an asset pool is now widely adopted by financial engineers to securitize various assets, including home morgages, automobile loans, credit card receivables, corporate loans, and defaultable bonds.

Tranching makes the securitization very flexible and allows financial intermediaries to design securities with highly heterogeneous characteristics that are "tailor-made" to meet diverse risk/return needs of potential investors. This flexibility naturally leads to the following optimal securitization problem: How can the issuer design securities that optimally make use of investors' heterogeneity? The goal of this paper is to study this problem. We capture investors' heterogeneity through their risk and time preferences as well as their endowments, and we do not limit the potential securities to debt and equity but include general limited liability securities, whose payoffs are backed solely by the given assets. An obvious question is whether reasonable assumptions lead to the optimality of tranching or some other simple securitization procedures.

To better illustrate our approach to the above research question, we describe the overall structure of our model as follows. The issuer possesses some assets, generating a cash flow X with maximal value $\bar{X}$ at time one. To raise capital at time zero, the issuer designs a basket of N limited liability securities, with N being less than or equal to the number of types of different investors in the market.[1] Issuing securities is costly, and we allow for both proportional and fixed issuing costs. The security i is a claim to a nonnegative payment $F_i = F_i(X)$, contingent on the realization of X. The securities

[1] As in Bolton and Scharfstein (1996), the issuer optimally decides, with which investors to trade, and so N may happen to be strictly smaller than the number of avalable investor types.

are backed solely by X, so that the total cash flow generated by all securities never exceeds X. To focus on the role of clientele effects arising from differences in preferences and endowments, we ignore the effects of asymmetric information and moral hazard and assume that the probability distribution of X is exogenously given and all market participants agree on it.[2]. Given the investors' heterogenous preferences, endowments and reservation utilities, the issuer knows the highest price that each investor is willing to pay for a particular security.

Our first main result is that the optimal securities can be described in terms of an endogenously determined optimal tranching. To design the optimal security, the issuer first optimally decides how much money to get from each of the investors. These amounts determine the investors' maximal marginal rate(s) of intertemporal substitution (MMRIS),[3] which captures the maximal marginal prices they are willing to pay at time zero for getting an infinitesimal part of X at time one. Then, the issuer ranks the investors according to the MMRIS of each investor and determines the optimal tranching of X with $X = \sum_j \text{Tranche}(Z_j, Z_{j+1})$. Here, $\text{Tranche}(a, b) = \max(a\,,\, \min(X, b)) - a$ is a standard tranche security, and the tranche thresholds, $0 = Z_{N+1} \le Z_N \le \cdots \le Z_1 \le Z_0 = \bar{X}$, are determined optimally given the investors' MMRIS.

The first very important decision that the issuer makes is whether to sell the super-senior standard debt $\text{Tranche}(0, Z_N)$. We show that the super-senior tranche is only sold if the retention costs are relatively high, so that there exists an investor whose MMRIS is larger than that of the issuer. If there are K investors whose MMRIS are larger

[2]Note that we do not need to require that the market participants know the true distribution of X, but rather that they have the same beliefs about it. In particular, the risk premium they require for buying securities could also be interpreted as an uncertainty premium the investors charge for not knowing the exact distribution of X

[3]We elaborate on this important concept in Section 5.

than that of the issuer, the standard debt security Tranche$(0, Z_{\mathrm{N-K+1}})$ is fully sold to these investors. Namely, the super-senior Tranche$(0, Z_N)$ is sold to the investor with the highest MMRIS, the second senior tranche is shared by the two investors with the highest MMRIS, the third senior tranche is shared by the three investors with the highest MMRIS, etc. The junior (equity) part of X, Tranche$(Z_{N-K+1}, \bar{X})$, is never fully sold. Namely, for every subsequent (in seniority) tranche inside this junior part, the issuer adds the next highest ranked investor to those who already hold senior tranches and shares this tranche with all these investors.

If the retention costs are low, the issuer retains the super-senior tranche and never fully sells any of the tranches. Then, the security design is similar to that in the previous paragraph: The issuer gradually sells parts of the tranches to the investors in the order of their decreasing MMRIS, so that the investor with the highest MMRIS gets a part in all tranches (except for the super-senior one), the investor with the second-largest MMRIS gets a part in all tranches except for the first two senior ones, and so on. Finally, the investor with the smallest MMRIS gets only a part of the most junior (equity) tranche, i.e., Tranche(Z_1, Z_0).

This prioritized tranche-sharing structure is very intriguing. It arises because of investors' risk aversion and the heterogeneity of their marginal valuations. The issuer optimally sells the most attractive senior tranche she wants to sell to the investor who values it the most. However, because this investor is risk averse, the marginal value of this tranche decreases with the level of cash flows. Precisely as the level of cash flow reaches the tranche threshold, the investor's marginal valuation reaches that of the second-highest ranked investor, and it is optimal for the issuer to sell a part of the subsequent tranche to this second-best investor. Continuing the process gradually, as the level of cash flow increases, investors with lower marginal valuations start getting parts of the relatively junior tranches, until the whole range of X is exhausted. Some of the junior tranches may collapse to one point $\bar{X}$, in which case the investor(s) with the lowest MMRIS do not get anything. Interestingly enough, when all market participants have exponential (i.e., constant absolute risk aversion (CARA)) utility functions, the risk sharing inside each tranche is linear, and optimal securities are given by portfolios of tranches.

The investors' MMRIS are shadow prices (i.e., the Lagrange multipliers characterizing this constraint-efficient allocation), and because of the structure described above, the problem of optimal securitization reduces to that of calculating the investors' MMRIS. Our second main result is that the investors' MMRIS can be calculated as a fixed point of an explicitly constructed contraction mapping. This result is crucial, both for numerical calculation of optimal securities and for studying the dependence of tranche thresholds on the parameters of the model. We use it to derive comparative statics and make several testable empirical predictions: (1) In most transactions initiated by an issuer, she sells multiple tranches but retains fractions of these tranches; (2) if the quality of the assets X is low and/or retention costs are high, the issuer sells the super-senior tranche; and (3) if the issuer is risk averse, the quality of assets X is high and/or the retention costs are low (i.e., the issuer's discount rate is low), the issuer retains the super-senior tranche, and the size of this non-securitized super-senior tranche is monotone increasing in the assets' quality and decreasing in the issuer's discount rate.

Even though our model is "an abstraction that gives insight rather than a realistic description of what we observe,"[4] we can test the model's predictions on the available transaction data. The tranche structure predicted by the model makes collaterized debt obligation (CDO) transactions prefectly suited for such a test.[5] CDOs are often clas-

[4] Allen and Gale (1989).

[5] A CDO transfers the credit risk of a pool of underlying collateral to investors by tranching the collateral cash flows into "tailor-made" securitized notes to meet the needs of different clientele.

sified based on different criteria. For example, based on underlying collateral, CDOs could be classified as collateralized bond obligations (CBOs), collateralized loan obligations (CLOs), etc. Based on issuance type, CDOs could be classified as cash flow CDOs and synthetic CDOs, and, based on motivation, CDOs could be classified as arbitrage CDOs and balance sheet CDOs. Until the recent crisis, CDOs were considered to be the fastest growing sector in the asset-backed security (ABS) market.[6] The global CDO issuance volume was estimated to be USD 157.14 billion in 2004, USD 271.8 billion in 2005, and USD 520.6 billion in 2006,[7] although in more recent years the issuance volume has decreased because of the financial crisis. Surprisingly, there is little academic literature devoted to theoretical or empirical studies of CDO transactions. We make use of a recent paper by Franke, Herrmann, and Weber (2007) that provides extensive empirical analysis of a large database of European CDO transactions. Franke, Herrmann, and Weber observe that: "information asymmetry is strong for collateralized loan obligations because loans are often given to small- or medium-sized firms whose identity is not revealed to investors. By contrast, in collateralized bond obligations (CBO), bond issuers are revealed and often are big firms or governments with publicly available information. Thus, for CBOs, the asymmetric information effect must be relatively small." Our model therefore should work better for CBO than CLO transactions. The most surprising finding of Franke, Herrmann, and Weber is that, in 54 percent of CBO transactions, the issuer retains the super-senior tranche. This is quite puzzling, given that most of the existing literature on security design predicts that the super-senior tranche should always be sold as it is the least informationally sensitive. Our model provides an explanation of this surprising phenomenon based on the risk aversion of the issuer. In fact, the three predictions 1-3 are confirmed by their findings. For prediction 1, Franke,

[6] Fabozzi, Davis, and Choudhry (2006), page 120.

[7] Securities Industry and Financial Markets Association Press release (2009-01-15)

Herrmann, and Weber find evidence that most transactions initiated by the issuer have three to five differently rated tranches. Based on our model, we conjecture that there are about five different investor classes whose clientele effects the issuers typically address. Furthermore, DeMarzo (2005) notes that "while intermediaries sell off many of these CMOs, they also retain significant fractions for their own portfolios," again in agreement with prediction 1. Predictions 2 and 3 are also confirmed by the findings of Franke, Herrmann, and Weber.[8] We conclude that, although the world economic crisis that started in 2007 has led many to critize securitization, our theory shows that the basic ideas of securitization are sound and could be used to improve the social welfare if carried out properly. As stated by Bernanke in a 2008 speech, "the ability of financial intermediaries to sell the morgages they originate into the broader capital market by means of the securitization process serves two important purposes: First, it provides originators much wider sources of funding than they could obtain through conventional sources, such as retail deposits; second, it substantially reduces the originator's exposure to interest rate, credit, prepayment, and other risks associated with holding mortgages to maturity, thereby reducing the overall costs of providing mortgage credit."[9] If in the future the counterparties seriously screen the underlying pool to reduce the issues of asymmetric information, our optimal security design model could be used by security issuers to efficiently raise cash and better serve those purposes.

[8]See Hypotheses 11, 12, and 13, for which they find strong empirical support. We discuss these findings in greater detail in Section 8. Franke, Herrmann, and Weber also test several other hypotheses. However, these hypotheses are related to asymmetric information and cannot be tested in our model.

[9]www.federalreserve.gov/newsevents/speech/bernanke20081031a.htm

2 Related Literature

The literature of optimal security design is vast. Probably the most closely related to ours is the paper by Allen and Gale (1989). They consider an economy, populated by firms and individuals. Each firm has several production plans and can issue *two* securities against each production plan. Issuing securities is costly. Individual investors cannot issue securities and also cannot short sell the firms' securities.[10] Furthermore, all investors know the prices of all securities that could potentially be issued in equilibrium. The issued securities are traded in a centralized market, and the prices are determined in a Walrasian equilibrium through market clearing.[11]

Allen and Gale show that an equilibrium always exists and all equilibrium allocations are Pareto-efficient. Their main result states that when the costs of issuing securities are fixed (i.e., the costs depend only on the number of issued securities and not on their nature), optimal securities are *extreme:* In any state of nature each security promises either the entire product of the firm or nothing. Firms optimally utilize the heterogeneity of investors' marginal valuations and design securities so that in every state all payoffs are allocated to the security held by the group that values it most.

The main difference between our model and that of Allen and Gale is that, in contrast to Allen and Gale, transactions in our model are over the counter (OTC) and there is no unique price for a given security. Each investor values a particular security in a different way, depending on his duration needs, capital requirements, and target risk-

[10] Or, equivalently, the cost of short selling is very high. Otherwise, short selling would be equivalent to issuing the security.

[11] In a subsequent paper, Allen and Gale (1991) consider a modification of their 1989 model in which short selling is allowed and there is a finite number of competing firms. They study efficiency of equilibrium allocations and do not characterize the nature of optimal securities.

return tradeoff.[12] Thus, the firm's objective is quite different in our model: Whereas in Allen and Gale (1989) the firm capitalizes on heterogeneous *marginal* valuations, in our model the firm uses differences in *true* valuations.[13] We also note that Allen and Gale assume that the firm does not get utility from future consumption, and therefore the payoff of the two securities is assumed to exhaust the production plan. For this reason, they cannot address the important empirical phenomenon that firms almost always retain a part of the assets.

The contribution of Allen and Gale is certainly of fundamental importance. However, since most of the ABS transactions are OTC, we believe that our model is better suited for studying the design of this class of securities. Furthermore, most real-world securities (e.g., tranches) have a monotone, continuous payoff. This empirical evidence stands in stark contrast to the optimality of extreme securities with discontinuous, potentially non-monotone payoffs, as is the case in Allen and Gale's model. It is also not clear how optimal securities can be calculated in their model if firms are allowed to issue more than two securities. By contrast, our model generates theoretical predictions that stand in agreement with real-world securitization. It predicts that optimal securities are *always monotone increasing and continuous* and explains why tranching is optimal and why the super-senior tranche is retained by the issuer in many transactions. Furthermore, our model generates a simple and efficient way to calculate optimal securities numerically and to derive comparative statics results.

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[12]Similarly to Allen and Gale, we model these differences in valuations through heterogeneity in preferences and endowments.

[13]Namely, in the Allen and Gale (1989) Walrasian equilibrium, the price of any traded security is uniquely determined and is the same for all market participants. However, due to short-selling constraints, each security is only held by the investor who values it highest at margin. By contrast, in our model, there is no predetermined price: Different investors are willing to pay different prices for the same security.

Another paper related to ours is Winton (1995). He studies the optimal securitization problem for a manager of a firm whose output can only be verified privately at a cost. The issuer optimally designs a basket of securities (contracts), one for each investor, and investors optimally choose the set of states in which they verify the firm's output. Surprisingly, even though all investors are identical, Winton shows that symmetric contracts (i.e., identical for all investors) are typically not optimal. Furthermore, when the manager and all investors are risk neutral, tranching is optimal: The manager places investors in an arbitrary order and then sells the tranches to the investors in the order of their seniority. The costly state verification model of Winton is certainly important for modeling leveraged buyouts and reinsurance contracts. However, it typically is not appropriate for modeling ABSs such as CBOs, where defaults (and hence the payoffs) are directly observable. Also, it is not clear from the results of Winton how the optimal tranche levels can be calculated and whether the results can be extended to the case when investors are risk averse. Winton notes that "the question of whether prioritized contracts are optimal when both manager and investors are risk averse remains open" (p.112). Our results are generally applicable and do not depend on the risk-neutrality assumption. They can be used to calculate optimal allocations as long as they are constrained efficient. In our model, investors are heterogeneous, and multiple tranches are optimally issued to address heterogeneous clientele. The ordering of investors by seniority is endogenously determined by their preferences and endowments. We also note that the number of creditors (i.e., investors to which the issuer sells securities) is endogenous in our model and is determined optimally by the issuer. This is indirectly related to the paper of Bolton and Scharfstein (1996).

A large part of the literature on security design focuses on the effects of asymmetric information. DeMarzo and Duffie (1999) develop a model in which the issuer has private information about the future payoff and signals a high value security by its willingness to retain a portion of the issue. They study the problem of ex ante security design: The issuer designs the security before obtaining a signal about its value. DeMarzo and Duffie show that, under certain conditions, the optimal ex ante security design is a standard debt. DeMarzo (2005) studies whether pooling and tranching is optimal for an informed security issuer. He shows that pooling may generate an information destruction effect for the issuer, in which case it is optimal to sell the assets separately. However, when the residual risk of each asset is not highly correlated, pooling and then selling a highly liquid and low-risk standard debt (senior tranche) is optimal because of a risk diversification effect. Both DeMarzo and Duffie (1999) and DeMarzo (2005) discuss the possibility of issuing multiple tranches. For example, DeMarzo writes, "If the issuer creates multiple tranches for an asset pool, then once the information is learned the issuer will choose a quantity of each tranche, or a tranche portfolio, to sell to investors." Since all investors are identical in his model, issuing multiple tranches is in fact equivalent to issuing a single security, whose payoff coincides with that of a portfolio of tranches. DeMarzo shows that, due to a concave liquidity effect, the payoff of the issuer is monotone increasing in the number of issued tranches, and as the number of issued tranches converges to infinity, the optimal tranche portfolio converges to the ex post optimal security design. Neither DeMarzo and Duffie (1999) nor DeMarzo (2005) show that issuing a finite tranche portfolio is an optimal security design. By contrast, we show that issuing portfolios of tranches is always optimal when all investors have CARA utilities and the optimal number of tranches could be strictly smaller than the number of investor types.

Gorton and Pennachi (1990) and Boot and Thakor (1993) show that, when both informed and uninformed investors are present in the market, it is optimal to split the asset into two securities: one senior and less informationally sensitive security, and one junior and more informationally sensitive security. Thus, their models have two types of investors, differing in their informational characteristics. They assume that the issuer fully sells the assets and do not address the optimal retention problem.

Fulghieri and Lukin (2001) and Axelson (2007) study the security design problem when outside investors have private information about the firm. They find that it is often optimal for the issuer to retain the senior tranche (the standard debt) and the optimal security may have a convex payoff. DeMarzo, Kremer, and Skrzypacz (2005) study security bid auctions in which privately informed bidders compete in an auction by bidding with securities, and they study the impact of moral hazard on the investors' optimal security design. Diamond (1993) analyzes optimality of short-term and long-term debt of different seniority. Hartman-Glaser, Piskorski, and Tchistyi (2009), Tchistyi (2009), and Piskorski and Tchistyi (2009) study optimal security design under moral hazard. None of these papers studies the effect of heterogeneous investor clientele on the securitization.

Some papers study the role of the value of control for a corporation and the agency costs in the management of the firm in the security design. See Harris and Raviv (1992) for a survey. Another large part of the literature is motivated by spanning risks. For surveys, see Allen and Gale (1994) and Duffie and Rahi (1995). Comprehensive surveys of the security design literature can be found in Harris and Raviv (1991) and in Allen and Winton (1995). Neither of these surveys discusses optimal securitization in OTC markets in the presence of heterogeneous clientele.

Several papers study implications of marketing costs for security design. For example, Madan and Soubra (1991) introduce marketing costs into the Allen and Gale (1989) model and show that the sharing of cash flow in several states may be optimal. Ross (1989) also explores the implications of marketing costs and shows that financial innovation can reduce the costs of marketing securities.

Several papers study the CDO design and structure of subordinated tranches empirically. See Duffie and Garleanu (2001), Mitchell (2004), Fender and Mitchell (2005), Franke, Herrmann, and Weber (2007), An, Deng, and Sanders (2008), and Franke and Weber (2009).

The remainder of the paper is organized as follows. Section 3 presents a formulation of the security design problem. Section 4 contains a complete solution to the security design problem when there is a single investor. In Section 5, we characterize optimal securities for an arbitrary number of investors. Section 6 shows how optimal tranche boundaries can be computed using the fixed point of a contraction mapping and provides several important comparative statics results. Section 7 presents a special case when all investors have CARA utilities, in which we show the optimality of standard tranche portfolios. Section 8 considers the case when the issuer is risk-neutral. Section 9 investigates the effects of fixed costs of issuing securities on security design. Section 10 concludes the paper and points out some future research directions.

3 The Problem of Security Design

The model's participants consist of an issuer and a set of N outside investors. The issuer owns assets that generate future cash flows given by a nonnegative bounded random variable X with $\text{esssup} X = \bar{X}$. In addition, the issuer is endowed with some other (not explicitly modeled) assets, generating a cash flow $(w_0\,,\,w_1)$. The issuer is an intertemporal expected utility maximizer, with von Neumann-Morgenstern utility u_S and a discount rate ρ_S

Investor $i\,,\,i=1,\cdots,N$ is endowed with an income flow $(w_{0i}\,,\,w_{1i})$. Each investor is an intertemporal expected utility maximizer, with a von Neumann-Morgenstern utility u_i and a discount rate ρ_i. All utilities are assumed to satisfy standard Inada conditions.

For simplicity, we assume that the cash flows (w_0, w_1) and $(w_{0i}\,,\,w_{1i}), i=1,\cdots,N$ are deterministic and exogenously given. Our results directly extend to the case when endowments are stochastic; however, the expressions become more complicated, and we omit it for the reader's convenience. In particular, we do not address security design, motivated by spanning risks in incomplete markets. We also do not consider the possibility of trading in other securities to reallocate capital between states and time periods and hedging a part of the risk inherent in X. Our techniques can be directly extended to allow for hedging and raising cash using bonds or other securities, and we leave it for future research.

It is important to note that the preference parameters $(\rho_S\,,\,u_S)$ and $(\rho_i\,,\,u_i)$ should not be interpreted directly as the "true" preferences of the investors. Rather, this is a stylized way to model the issuer's and the investor's subjective attitude to the particular sources of risk and return of the assets X. For example, if the discount rate ρ_S is relatively high, the issuer (e.g., a financial intermediary, such as a bank or a credit card company) faces a high cost of holding the assets (equivalently, the issuer has highly profitable investment opportunities), and so the issuer's demand for funds is high. Alternatively, the discount rate may be high because the issuer may face credit constraints or, as for many banks and others in the financial services industry, binding minimum capital requirements.[14] Similarly, investors may have different demands for assets (e.g., with different duration and risk). An investor's risk aversion, determined by u_i, can be interpreted as the size of the risk premium the investor requires for holding the particular type of risk, inherent in X.

To raise cash, the issuer creates a basket $F_i\,,\ i = 1, \cdots, N$ of limited-liability securities backed by the asset X.[15] We assume that there is no asymmetric information and

[14]By placing the loan portfolio in a so-called special purpose vehicle (SPV) and selling the tranches to investors, a bank can remove that portion of its loan portfolio from its balance sheet and thereby expand its lending capacity.

[15]Note that F_i could be interpreted as a portfolio of Arrow-Debreu like securities with the continuous state space $[0, \bar{X}]$. Nevertheless, we refer to F_i as a security for simplicity. Later on in the paper, we will show that with exponential utility functions, F_i is actually a linear combination of some basic securities:

therefore the true probability distribution of the payoff $F_i \geq 0$ of the security i is known to all market participants. The claims of security holders are secured solely by the asset X, and therefore the basket satisfies:

$$F = \sum_{i=1}^{N} F_i \leq X.$$

Given a security design (F_i), the issuer retains the residual cash flow $X - F$.

We assume that the issuer is a monopolist. He offers a security F_i to investor i, and the investor offers him the price $P_i = P_i(F_i)$. Issuing securities is costly, with both fixed and variable costs for each transaction. For example, there are legal fees and accounting fees at inception at the securitization level. At the tranche level, there are underwriting and rating agency fees.[16] For simplicity, we assume in our model that there is a variable cost C_i of issuing a security i being proportional to its price, $C_i = \alpha P_i$ for some $\alpha \in (0, 1)$. In Section 9, we introduce fixed issuing costs on top of the proportional costs.

As is common in the optimal contracting literature, we assume that an investor is willing to take any contract (P_i, F_i) satisfying the investor's participation constraint

$$u_i(c_{0i}) + e^{-\rho_i} E[u_i(c_{1i})] \geq L_i, \tag{1}$$

where

$$c_{0i} = w_0 - P_i , \; c_{1i} = w_{1i} + F_i(X)$$

tranches, which are anagolous to the Arrow-Debreu securities.

[16]We thank Sean Reddington and Steve Parsons from BNP Paribas for useful comments on the cost of securitization.

is the investor's consumption after entering the contract, and

$$L_i \;=\; u_i(w_{0i}) \;+\; e^{-\rho_i}\, u_i(w_{1i})$$

is the investor's reservation utility, equal to the utility before entering the contract.[17] Given the contracts $(P_i\,,\,F_i)\,,\; i \,=\, 1\,,\,\cdots\,,\,N$, the issuer's consumption is given by:

$$c_{0S} \;=\; w_0 \,+\, (1 \,-\, \alpha)\sum_{i=1}^{N} P_i, \qquad c_{1S} \;=\; w_1 \,+\, X \,-\, F(X)\,. \tag{2}$$

The issuer's securitization problem is to design the basket (F_i) so as to maximize his utility,

$$u_S(c_{0S}) \;+\; e^{-\rho_S}\, E\big[\, u_S(c_{1S})\,\big],$$

under the budget constraints (2) and participation constraints (1).

Because the issuer may optimally decide to retain a part of the assets used to back the security payoffs, her problem does not simply reduce to maximizing the amount she receives from the investors. Rather, the problem for the issuer is to allocate the payoffs of the securities across states of the world to maximize her expected utility from both selling securities and retaining part of the assets. If the quality of the underlying assets is high and the retention cost of keeping a part of the assets is low, the issuer might be willing to retain a large part of them. However, the investors will also pay a higher price for a given slice of these better quality assets, increasing the issuer's incentive to sell. Different risk attitudes and time preferences also generate a similar tradeoff.

Finally, we note that, by monopolist assumption, participation constraints (1) are

[17] The assumption that the reservation utility coincides with the utility before entering the contract is made for technical purposes, to avoid discontinuities in the price P_i. It is possible to relax this assumption at the cost of getting messier results.

always binding for the investors, and therefore the trade will always happen at the price:

$$P_i(F_i) \;=\; w_{0i} \;-\; v_i(L_i \;-\; e^{-\rho_i}\, E[u_i(w_{1i} + F_i(X))])\,, \tag{3}$$

where v_i is the inverse of the investor's utility,

$$v_i(u_i(x)) \;=\; x.$$

4 Single Investor

In this section we present a solution to the optimal security design problem for the case of a single investor. Since there is only one investor $i = 1$, we use index B to denote the corresponding characteristics (w_{0B}, w_{1B}), ρ_B and u_B.

If the constraints $0 \le F(X) \le X$ are not binding, risk sharing between the issuer and the investor is Pareto-efficient and maximizes social welfare:

$$\max_F \,\big(\, u_S(w_1 + X - F) \;+\; a\, u_B(w_{1B} \;+\; F)\,\big)$$

for some welfare weight $a > 0$. Writing down the first-order condition, we get that $F(x) = g(a, x)$ where the function $g(a, x)$ is the unique solution to:

$$a\, u_B'(w_{1B} \;+\; g) \;-\; u_S'(w_1 + x \;-\; g) \;=\; 0\,. \tag{4}$$

By definition, our allocation is constrained Pareto-efficient, and therefore the function (4) describes the optimal allocation for those values of X for which the constraints are not binding, and $F(X) = 0$ or X when the corresponding constraint is binding. Below, we provide necessary and sufficient conditions for the constraints to be binding and describe the corresponding regions of values of X.

To this end, we need several definitions. Let

$$P_{\max} \;=\; w_{0B} \;-\; v_B\,(L_B \;-e^{-\rho_B}\,E[u_B(w_{1B}+X)])$$

be the price that the investor is willing to pay for getting the whole X (that is, when the constraint $F(X) \le X$ is binding for all values of X), and let

$$P_{\mathrm{mid}} \;=\; w_{0B} \;-\; v_B\,\left(L_B \;-e^{-\rho_B}\,E\left[u_B\left(w_{1B}+g\left(\frac{u'_S(w_1)}{u'_B(w_{1B})}\;,\;X\right)\right)\right]\right)$$

be the price of the security that the investor would get if the constraints $0 \le F \le X$ were not binding and $a = u'_S(w_1)/u'_B(w_{1B})$. Then, define

$$\begin{aligned} K_{\max} \;&=\; \log\frac{u'_S(w_1)/u'_S\,(w_0\;+\;(1-\alpha)\,P_{\max})}{(1-\alpha)\,u'_B(w_{1B}+\bar{X})/u'_B(w_{0B}-P_{\max})}, \\ K_{\mathrm{mid}} \;&=\; \log\frac{u'_S(w_1)/\,u'_S\,(w_0\;+\;(1-\alpha)\,P_{\mathrm{mid}})}{(1-\alpha)\,u'_B(w_{1B})/u'_B(w_{0B}\;-\;P_{\mathrm{mid}}))}, \quad \text{and} \\ K_{\min} \;&=\; \log\frac{u'_S(w_1+\bar{X})/u'_S\,(w_0)}{(1-\alpha)\,u'_B(w_{1B})/u'_B(w_{0B})}\,. \end{aligned} \tag{5}$$

All three numbers, $K_{\max}$, K_{mid} , $K_{\min}$, are given by differences between the growth rates of the undiscounted marginal values of consumption of the issuer and the investor. $K_{\max}$ corresponds to the case when the investor gets the whole X and is calculated at the maximal level $\bar{X}$. Similarly, K_{mid} corresponds to the case when the constraints $0 \le F \le X$ are not binding and is calculated at $X = 0$. Finally, $K_{\min}$ corresponds to the case when the constraint $F \;\ge\; 0$ is binding for all values of X so that there is no trade ($F = 0$), and the constraint is calculated at the maximal level $X = \bar{X}$. It follows directly from the definition that

$$K_{\max} \;>\; K_{\mathrm{mid}} \;>\; K_{\min}\,.$$

We also need the inverse of the investor's marginal utility I_B and the inverse of the issuer's marginal utility I_S:

$$I_B(u'_B(x)) = x$$

$$I_S(u'_S(x)) = x\,.$$

Under the conditions described, we can state:

Theorem 4.1 *(1) If*

$$\rho_S \;-\; \rho_B \;>\; K_{\max},$$

then full selling is optimal,

$$F(X) \;=\; X\,;$$

(2) if

$$K_{\max} \;>\; \rho_S \;-\; \rho_B \;>\; K_{\text{mid}},$$

then there exists a threshold $Z(a) \,\in\, (0\,,\,\bar{X})$ such that $F(X)$ is a combination of a standard debt and Pareto-optimal sharing (4) for $X \,\geq\, Z(a)$,

$$F_a(X) \;=\; \begin{cases} X & ,\, X \,\leq\, Z(a) \\ g(a,X) & ,\, X \,>\, Z(a) \end{cases}\;;$$

(3) if

$$K_{\text{mid}} \;>\; \rho_S - \rho_B \;>\; K_{\min},$$

then there exists a threshold $Z(a) \,\in\, (0\,,\,\bar{X})$ such that the issuer retains the part of X below $Z(a)$ and optimally shares risk according to (4) for $X \,\geq\, Z(a)$,

$$F_a(X) \;=\; \begin{cases} 0 & ,\, X \,\leq\, Z(a) \\ g(a,X) & ,\, X \,>\, Z(a) \end{cases}\;;\quad \textit{and}$$

(4) if

$$K_{\min} > \rho_S - \rho_B,$$

then there is no trade, that is, $F(X) = 0$.

Furthermore,

$$Z(a) = \begin{cases} I_B(a^{-1}u_S'(w_1)) - w_{1B} & \text{in case (2)} \\ I_S(a\, u_B'(w_{1B})) - w_1 & \text{in case (3)} \end{cases},$$

and a *is the unique solution to*

$$a = \frac{(1-\alpha)\, e^{\rho_S}\, u_S'\big(w_0 + (1-\alpha)\, P_B(F_a(X))\big)}{e^{\rho_B}\, u_B'\big(w_{0B} - P_B(F_a(X))\big)}, \tag{6}$$

where P_B *is given by* (3).

There are several interesting features of the optimal security that are different from those typically present in a model with asymmetric information and risk-neutral investors. First, the optimal security F is always monotone increasing. In contrast, in asymmetric information models, the optimal security is not necessarily monotone increasing, and monotonicity of F has to be imposed as an additional constraint. See DeMarzo and Duffie (1999).

Furthermore, in contrast with most of the existing literature on optimal security design (see, e.g., DeMarzo and Duffie (1999), DeMarzo (2005), and Hartman-Glaser, Piskorski, and Tchistyi (2009)), we do not need to require that the discount rate of the issuer be larger than that of the investor. When both the issuer and the investor are risk averse and the essential supremum $\bar{X}$ of X is sufficiently large, the trade will always take place, even if ρ_S is much smaller than ρ_B.

To gain a better understanding of the intuition behind the optimal risk sharing of

Theorem 4.1, let us calculate the maximal marginal price π_B the investor is willing to pay at time zero for an additional unit of optimal consumption at time one. Clearly, π_B is determined by the equality of the marginal utility loss $\pi_B\, u_B'(c_{0B})$ and the marginal utility gain $e^{-\rho_B}\, u_B'(c_{1B})$:

$$\pi_B\, u_B'(c_{0B}) \;=\; e^{-\rho_B}\, u_B'(c_{1B}) \;\Leftrightarrow\; \pi_B \;=\; \frac{e^{-\rho_B}\, u_B'(c_{1B})}{u_B'(c_{0B})}\,. \tag{7}$$

That is, π_B is equal to the marginal rate of intertemporal substitution (MRIS) of the investor.[18]

Similarly, the minimal marginal price π_S the issuer is willing to accept at time zero in exchange for losing a unit of consumption at time one is given by:

$$\pi_S \;=\; \frac{e^{-\rho_S}\, u_S'(c_{1S})}{(1-\alpha)\, u_S'(c_{0S})}, \tag{8}$$

which is equal to the MRIS of the issuer.[19] The trade will only take place if the investor is willing to pay more than the issuer is willing to accept:

$$\pi_B \;\geq\; \pi_S\,. \tag{9}$$

[18]Note that, in a Walrasian equilibrium, the first-order conditions for an investor B holding an asset with payoff D at time one imply that the price P_0 of this asset at time zero is given by

$$P_0 \;=\; E[\pi_B\, D].$$

For this reason, π_B is commonly referred to as the the pricing kernel, or stochastic discount factor (see Duffie (2001)). In our model, π_B only determines the *marginal value* of the payoff and not its true value, which may be different across investors.

[19] Here, the factor $(1-\alpha)$ appears in the denominator because of the issuing costs. However, we still refer to this quantity as the MRIS for convenience.

In general, the inequality $\pi_B \geq \pi_S$ will be violated for some levels of X. If the cost of holding assets is very high, ρ_S will be so large that the issuer will be willing to sell the whole X to the investor (case (1)). Since u_S and u_B are concave, the marginal consumption values u'_S and u'_B are decreasing in the level of X. If ρ_S is high, but not too large (case (2)), inequality (9) will hold for small values of X. However, as X becomes sufficiently large, the marginal benefit $u'_B(c_{1B})$ from holding the security F becomes too small, the investor is not willing to pay a high enough price for getting high levels of X, and it is optimal for the issuer to retain a part of the upper tail of X.

If the discount rate ρ_S is relatively low (case (3)), the marginal benefit of the issuer from holding the lower (less risky) part of X is high, and it is optimal for the issuer to retain it. However, as the level of X increases, the marginal benefit starts going down until it becomes so small that the issuer is willing to sell part of it to the investor. Finally, if the discount rate ρ_S is very small (case (4)), the marginal benefit of X is so high that the issuer retains the whole X.

Differentiating (4), it is possible to show (see, e.g., Wilson (1968)) that:

$$\frac{d}{dx}\,F_a(X) \;=\; \frac{R_B(c_{1B})}{R_B(c_{1B})\,+\,R_S(c_{1S})}, \tag{10}$$

where

$$R_K(x) \;=\; -\frac{u'_K(x)}{u''_K(x)}\;,\;\; K \;=\; B\,,\,S.$$

That is, the slope of the optimal security is determined by the investor's risk tolerance relative to that of the issuer. When the investor's risk tolerance is high, the investor will be willing to take more risk for a smaller price, and the slope will be close to 1. However, if the issuer's risk tolerance is high, the issuer will be willing to hold more risk, and therefore the slope of F_a will be small. In the extreme case when the issuer is risk neutral, the slope converges to zero, and we arrive at the following proposition.

Proposition 4.2 *Suppose that the issuer is risk neutral. Then, the optimal security is a standard debt,*

$$F(X) \;=\; \min(X\,,\, d)$$

for some $d \geq 0$.

It is interesting to note that standard (risky) debt security $\min(X, d)$ is also optimal for a large class of asymmetric information models when both the investor and the issuer are risk neutral. See DeMarzo and Duffie (2009). Proposition 4.2 suggests that it is the risk neutrality of the issuer that is responsible for this simple form of optimal security. In the framework of mortgage and other loan securitizations, standard debt is also known as the single tranche CDO. The face value d of the debt is often referred to as the first loss position. We come back to these issues in Section 7.

Finally, we also note that the result of Theorem 4.1 is related to the literature of optimal insurance design, in particular to the optimal insurance contract derived by Raviv (1979). Raviv considered the following problem: The insured wants to buy insurance against a random claim X. He designs an $F(X) \in [0, X]$, and the insurer agrees to pay him $F(X)$ in exchange for payment P, determined through the insurer's reservation utility. Raviv showed that the optimal insurance contract $F(X)$ is characterized by an expression, similar to that of Theorem 4.1: $F(X) = 0$, or X for X below a threshold Z, and $F(X)$ is given by the Pareto-optimal sharing rule (4) for X above the threshold.

Raviv's setup differs from ours in several important aspects: (a) the agent is buying insurance against *losses* X, whereas in our setting the issuer is selling a part of the profits X; (b) Raviv considers a zero period model, so there is no discounting, and the issue of raising cash does not arise; (c) insurance is costly for the insurer, with the cost being a function $c(F)$. Raviv shows that the optimal contract is characterized by a deductible (i.e., case (3) of Theorem 4.1 takes place) if and only if the cost c of insurance depends on the insurance payment. In contrast to Raviv's result, we show that, due to the multi-period nature of our model, the structure of the optimal contract is determined by the MRIS of the investor and the issuer, i.e., (7) and (8). Furthermore, Raviv does not show when each of the four cases of Theorem 4.1 occurs and does not provide any expression for the threshold Z of the deductible. By contrast, we explicitly characterize when each of the regimes (1) through (4) occurs and provide a closed-form expression for the threshold $Z(a)$.

5 Heterogeneous Investors

In this section we extend Theorem 4.1 to the case of multiple heterogeneous investors. To understand why heterogeneity is important, let us first examine the case when all investors are risk neutral. In that case investor i is willing to pay

$$P_i(F_i) \;=\; e^{-\rho_i}\, E[F_i(X)]$$

for a security F_i. Therefore, diversifying between different investors is never optimal for the issuer. The investor with the lowest discount rate will always be willing to pay the highest price, and the issuer will always sell the whole $F = \sum_i F_i$ to this investor because the pricing is linear. Namely,

$$\sum_i P_i(F_i) \;\leq\; e^{-\rho_{\min}} \sum_i E[F_i]\,.$$

Thus, with risk neutrality, heterogeneity does not play any role for securitization. However, when investors are risk averse, the situation is completely different: Every investor gets a non-zero part of X if the the maximal payoff $\bar{X}$ is sufficiently large.

Proposition 5.1 *The following are true:*

- *If all investors are risk neutral, then only the investor with the lowest discount rate* *will participate in a trade.*

- *If investors are risk averse and $\bar{X}$ is sufficiently large, then all investors investors will get a non-zero part of X.*

The drastic difference between the risk-neutral case and the risk-averse case arises because the marginal value of an additional unit of X for a risk-averse investor is monotone decreasing with the level of X. Suppose, for example, that there are two investors, 1 and 2, and the discount rate ρ_1 of investor 1 is much lower than that of investor 2. Then, clearly, investor 1 typically is willing to pay more for a security than investor 2. However, as the level of X becomes sufficiently high, investor 1's MRIS decreases and eventually becomes smaller than that of investor 2. It therefore becomes optimal for the issuer to sell a part of the high-risk portion of X to investor 2.

It turns out that the nature of the optimal allocation is uniquely characterized by the investors' maximal marginal rates of intertemporal substitution (MMRIS)

$$Y_i \;=\; \frac{e^{-\rho_i}\, u_i'(w_{1i})}{u_i'(c_{0i})}$$

and the issuer's MMRIS

$$Y_S \;=\; \frac{e^{-\rho_S}\, u_S'(w_1)}{(1-\alpha)\, u_S'(c_{0S})}\,.$$

Here,

$$c_{0i} \;=\; w_{0i} \;-\; P_i(F_i) \qquad \text{and} \qquad c_{0S} \;=\; w_0 \;+\; (1-\alpha)\sum_i P_i(F_i).$$

Note that fixing Y_i is equivalent to fixing the prices $P_i(F_i)$ the investors pay to the issuer. Given the optimal allocation (F_i), we assign rankings to investors according to the following definition.

Definition 5.2 *For an investor i, we denote by* $\mathrm{rank}(i)$ *the number that the investor will have when all investors are reordered so that larger* $\mathrm{rank}(i)$ *implies larger Y_i. That* *is,* $\mathrm{rank}(i) = N$ *if investor* i *has the largest* Y_i, $\mathrm{rank}(i) = N-1$ *if investor* i *has the second largest* Y_i*, etc., and* $\mathrm{rank}(i) = 1$ *if investor* i *has the smallest* Y_i.[20]

Furthermore, we denote by J *the number of investors for which* Y_i *is smaller than* Y_S.

Having defined the ranking order, we can define the tranche thresholds.

Definition 5.3 *For each* $i = 1, \cdots, N$*, let:*

$$a_i \stackrel{def}{=} \frac{e^{\rho_S}\,(1-\alpha)\,u_S'(c_{0S})}{e^{\rho_i}\,u_i'(c_{0i})} = \frac{Y_i\,e^{\rho_S}\,(1-\alpha)\,u_S'(c_{0S})}{u_i'(w_{1i})}. \tag{11}$$

Fix $k \in \{0, 1, \cdots, N, N+1\}$.

- *For* $k = 0$ *we define*

$$Z_0 = \bar{X}.$$

- *For* $1 \leq k \leq J$*, let* $K = \mathrm{rank}^{-1}(k)$ *be the investor whose rank is equal to* k *and*

$$\tilde{Z}_k = I_S(a_K\,u_K'(w_{1K})) - w_1 + \sum_{i\,:\,\mathrm{rank}(i) \geq k+1} \left(I_i\left(a_i^{-1}\,a_K\,u_K'(w_{1K})\right) - w_{1i}\right) \tag{12}$$

and

$$Z_k = \min\{\bar{X}\,,\,\tilde{Z}_k\}.$$

- *For* $k = J+1$ *we define:*

$$\tilde{Z}_{J+1} = \sum_{i\,:\,\mathrm{rank}(i) \geq J+1} \left(I_i\left(u_S'(w_1)\,a_i^{-1}\right) - w_{1i}\right), \tag{13}$$

[20] If two investors have the same Y_i, we give them subsequent rankings in any order. However, it is important to do it so that $\mathrm{rank}(i) \neq \mathrm{rank}(j)$ for $i \neq j$.

and

$$Z_{J+1} \;=\; \min\{\bar{X}\,,\,\tilde{Z}_{J+1}\}.$$

- *For* $J+2 \le k \le N$, *let* $K \;=\; \mathrm{rank}^{-1}(k-1)$ *and*

$$Z_k \;=\; \sum_{i\,:\,\mathrm{rank}(i)\ge k} \left(I_i\left(a_i^{-1}\,a_K\,u_K'(w_{1K})\right)\;-\;w_{1i}\right)\,, \tag{14}$$

and

$$Z_k \;=\; \min\{\bar{X}\,,\,\tilde{Z}_k\}.$$

- *For* $k = N+1$, *we define* $Z_{N+1} = 0$.

A direct calculation (see Appendix) implies that:

$$Z_0 \;\ge\; Z_1 \;\ge\; \cdots \;\ge\; Z_N \;\ge\; Z_{N+1} \;=\; 0.$$

The security,

$$\mathrm{Tranche}(a,b) \;=\; \begin{cases} 0 & ,\; x < a \\ x-a & ,\; x \,\in\, (a,b) \\ b-a & ,\; x \,>\, b \end{cases},$$

will be referred to as a tranche. For simplicity, we denote:

$$\mathrm{Tranche}_j \;=\; \mathrm{Tranche}(Z_{j+1}, Z_j).$$

Note that $\mathrm{Tranche}_j$ will be empty if $Z_j = Z_{j+1}$. We say that an investor i participates in the tranche, $\mathrm{Tranche}_j$, if $F_i(x) \not\equiv 0$ for $x \in (Z_{j+1}, Z_j)$.

We are now ready to state the main result of this section.

Theorem 5.4 *There always exists a unique optimal allocation* $\{F_i\}_{i=1}^N$*. It is non-zero (i.e.,* $F(X) \not\equiv 0$*) if and only if:*

$$\frac{e^{-\rho_S}\, u_S'(w_1+\bar{X})}{(1-\alpha)\, u_S'(w_0)} \;\le\; \max_i \frac{e^{-\rho_i}\, u_i'(w_{1i})}{u_i'(w_{0i})}. \tag{15}$$

If (15) holds, then the following is true:

(1) Optimal securities F_i *and the retained part* $X - F(X)$ *are continuous and (weakly) monotone increasing in* X*;*

(2) If $Y_i < Y_S$*, then the investor* i *only participates in tranches* Tranche_j *with indices* $j \le \text{rank}(i) - 1$*;*

(3) If $Y_i \ge Y_S$*, then the investor* i *only participates in tranches* Tranche_j *with indices* $j \le \text{rank}(i)$*;*

(4) The issuer fully sells the part of X *below* Z_{J+1} *and retains a part of* X *for* $X > Z_{J+1}$*. That is,*

$$F(X) \;=\; \sum_i F_i(X) \;=\; X$$

if $X \le Z_{J+1}$ *and* $F(X) < X \Rightarrow F(X) < X$ *otherwise;*

(5) For each Tranche_j*, there exists a function* $\mu_j(X)$ *such that:*

$$\frac{e^{-\rho_i}\, u_i'(c_{1i})}{u_i'(c_{0i})} \;=\; \mu_j(X) \tag{16}$$

for all investors i *participating in* Tranche_j*. Furthermore,*

(a) If $j \ge J+1$*, then:*

$$\mu_j(X) \;>\; \frac{e^{-\rho_S}\, u_S'(c_{1S})}{(1-\alpha)\, u_S'(c_{0S})} \quad \textit{and}$$

(b) If $j < J+1$, then:

$$\mu_j(X) \;=\; \frac{e^{-\rho_S}\, u_S'(c_{1S})}{(1-\alpha)\, u_S'(c_{0S})}. \tag{17}$$

First, we note that Equation (16) and Equation (17) uniquely determine the allocation. Indeed, substituting $c_{1i} = w_{1i} + F_i(X)$ into (16) gives $F_i \;=\; I_i(\mu_j(X)\, u_i'(c_{0i})\, e^{\rho_i}) - w_{1i}$. Then, for $j \;\leq\; J+1$, the function μ_j is uniquely determined by the constraint $\sum_i F_i(X) \;=\; X$, and for $j \geq J$, the function μ_j is uniquely determined by (17) and the issuer's budget constraint $c_{1S} \;=\; w_0 \;+\; X \;-\; \sum_i F_i(X)$.

As we have explained above, the structure of the optimal securitization is determined by the investors' MRIS (16). Since both the issuer's consumption $c_{1S} \;=\; w_1 + X - F(X)$ and the investors' consumption $c_{1i} \;=\; w_{1i} \;+\; F_i(X)$ are monotone increasing in X, the marginal values $u_S'(c_{1S})$ and $u_i'(c_{1i})$ of the issuer's and investors' consumption are monotone decreasing in X. If the MMRIS of all investors are smaller than that of the issuer,[21] then the super-senior tranche Tranche_N is not sold at all. Consequently, for $X \in [0, Z_N]$, MRIS of each investor i stays constant (equal to the MMRIS Y_i), whereas the MRIS of the issuer is monotone decreasing with the value of X. By contrast, if there is at least one investor i whose MMRIS Y_i is larger than that of the issuer[22], then the investor with the highest rank N holds the whole super-senior tranche, $\text{Tranche}_N \;=\; [0, Z_N]$. Furthermore, since $F(X) = X$ for all $X \in [0, Z_{J+1}]$, the MRIS of the issuer stays constant (equal to Y_S) when X varies in this interval, whereas the MRIS of investors with ranks higher than J are monotone decreasing with the value of X. Finally, for $X \;\geq\; Z_{J+1}$, both the MRIS of the issuer and the MRIS of all investors that already hold a part of X are monotone decreasing in X. Thus, two things happen: The issuer is willing to sell higher-risk portions of X at a lower price, and the investors that already own a part of

[21]That is, $\max_j Y_j \;<\; Y_S$

[22]That is, $\max_j Y_j \;>Y_S$.

the senior tranches are not willing to pay a high enough price for an additional part of X. These two effects become strong enough precisely when the level of X crosses the corresponding threshold Z_i, and the issuer then sells a part of X above this threshold to the next investor in the hierarchy, determined by the rank of MMRIS.

We illustrate the structure of optimal securities in the following example.

Example. Suppose that there are three investors with

$$Y_1 < Y_S < Y_2 < Y_3 \, .$$

Then, $J = 1$ and

$$\bar{X} \;=\; Z_0 \;>\; Z_1 \;>\; Z_2 \;>\; Z_3 \;>\; Z_4 = 0$$

if $\bar{X}$ is sufficiently large. In this case, optimal securities have the following structure:

- For $x \leq Z_3$, $F_3(x) = x$, so investor 3 gets the whole super-senior tranche;
- For $x \in [Z_2, Z_3]$, $F_2, F_3 > 0$ and $F_2 + F_3 = X$, so investors 2 and 3 share the full pie;
- For $x \in [Z_1, Z_2]$, investors 2 and 3 still share the pie, but the issuer retains a part of it: $F_1 = 0$, $F_2, F_3 > 0$ and $F_2 + F_3 < X$; and
- Finally, for $x > Z_1$, $F_1, F_2, F_3 > 0$ and $F_1 + F_2 + F_3 < X$.

In particular, Tranche$_1$ is a boundary between the full selling regime $[0, Z_2]$ and the partial selling regime $[Z_1, \bar{X}]$ with all investors participating. Investor 1 starts participating in the tranches "with delay," only after the intermediate (mezzanine) tranche $[Z_2, Z_1]$ is (partially) sold to investors 2 and 3.

It is well known (see, e.g., Borch (1962)) that, in an *unconstrained* Pareto-efficient allocation, marginal utilities of all agents are *co-linear for all values of* X. Since our allocation is by construction *constrained* Pareto-efficient, Borch's result is not true anymore. However, Theorem 5.4 (Equation (16)) implies that, for each fixed tranche, the investors participating in this tranche do share it in a Pareto-efficient way.

Recall that

$$R_i(x) = -\frac{u_i'(x)}{u_i''(x)}$$

is the absolute risk tolerance of investor i. Wilson (1968) showed that the slopes of sharing rules in a Pareto-efficient allocation can be characterized in terms of agents' absolute risk tolerances. The following result is an extension of Wilson's characterization for the constrained Pareto-efficient allocation in our model.

Proposition 5.5 • *For an investor i with* $\text{rank}(i) \geq J+1$,

$$\frac{d}{dx}F_i(x) = \begin{cases} 0, & x \leq Z_{\text{rank}(i)+1} \\ \frac{R_i(c_{1i})}{\sum_{j\,:\,\text{rank}(j)\geq k} R_j(c_{1j})}, & x \in (Z_{k+1}, Z_k),\, J+1 \leq k \leq \text{rank}(i) \\ \frac{R_i(c_{1i})}{R_S(c_{1S}) + \sum_{j\,:\,\text{rank}(j)\geq k+1} R_j(c_{1j})}, & x \in (Z_{k+1}, Z_k),\, 0 \leq k < J+1 \end{cases};$$

• *For an investor i with* $\text{rank}(i) \leq J$,

$$\frac{d}{dx}F_i(x) = \begin{cases} 0, & x \leq Z_{\text{rank}(i)} \\ \frac{R_i(c_{1i})}{R_S(c_{1S}) + \sum_{j\,:\,\text{rank}(j)\geq k+1} R_j(c_{1j})}, & x \in (Z_{k+1}, Z_k),\, 0 \leq k \leq \text{rank}(i)-1 \end{cases}.$$

The intuition behind the formulae for the slope is the same as the one for formula (10): The fraction of the aggregate risk the investor i ends up taking is proportional to his risk tolerance. In particular, investors with high risk tolerance (low risk aversion) generally are willing to pay more for the securities and, consequently, end up getting a larger piece of the pie.

6 Fixed-Point Equation and Comparative Statics

By Theorem 5.4, the optimal allocation is uniquely determined as soon as we know the rank of every investor, as well as the thresholds Z_k. By Definitions 5.2 and 5.3, both the ranks and the thresholds are uniquely determined by the N-tuple of numbers (a_i). Given the N-tuple (a_i), we denote $b_i = a_i^{-1}$ as their reciprocals and denote by $\mathbf{b} = (b_i)$ the vector of these reciprocals. We denote by $(Z_i(\mathbf{b})\,,\, i = 0, \cdots, N+1)$ the corresponding thresholds and by $(F_i(\mathbf{b})\,,\, i = 1, \cdots, N)$ the corresponding allocation. By definition (see (11)), the optimal allocation must satisfy:

$$b_i \;=\; \frac{e^{\rho_i}\, u_i'(c_{0i})}{(1-\alpha)\, e^{\rho_S}\, u_S'(c_{0S})} \;=\; \frac{e^{\rho_i}\, u_i'\,(w_{0i} - P_i(F_i(\mathbf{b})))}{(1-\alpha)\, e^{\rho_S}\, u_S'\Big(w_0 \,+\, (1-\alpha)\, \sum_j\, P_j(F_j(\mathbf{b}))\Big)}$$

for all $i = 1, \cdots, N$. This is a highly non-linear system of equations for the vector $\mathbf{b}$. It is by no means clear how to solve it analytically or even numerically and how the solution would depend on the microeconomic characteristics of the model.

In this section we show that this N-tuple is the unique fixed point of a contraction mapping defined on an explicitly given compact set and therefore can be easily calculated by successive iterations.

We use the common notation b_{-i} to denote the vector of all coordinates of $\mathbf{b}$ except for b_i.

Lemma 6.1 *For each $i = 1, \cdots, N$, there exists a unique, piecewise C^1 function*

$$H_i \;=\; H_i(C, b_{-i})$$

solving

$$H_i(C, b_{-i}) \;=\; e^{\rho_i}\, C\, u_i'\,\big(w_{0i} \;-\; P_i\big(F_i\big(X, \big(H_i(C, b_{-i}), b_{-i}\big)\big)\big)\,\big)\,. \tag{18}$$

The function H_i is monotone increasing in C and b_{-i}, and $C^{-1}\, H_i$ is decreasing in C.

Now, we are ready to formulate the main result of this section. To this end we need some definitions. Let

$$P_i^{\max} \;=\; w_{0i} \;-\; v_i\left(L_i - e^{-\rho_i}\, E[u_i(w_{1i} + X)]\right)$$

be the price that the investor i will pay for the whole X,[23]

$$C_{\min} \;=\; \left(e^{\rho_S}\, u_S'\,(w_0)\right)^{-1} \;\;,\;\; C_{\max} \;=\; \left((1-\alpha)\, e^{\rho_S}\, u_S'\left(w_0 \;+\; (1-\alpha)\,\sum_i\, P_i^{\max}\right)\right)^{-1}$$

and

$$\beta_i^{\max} \;=\; \log\left(C_{\max}\, e^{\rho_i}\, u_i'\,(w_{0i} - P_i^{\max})\right) \;\;,\;\; \beta_i^{\min} \;=\; \log\left(C_{\min}\, e^{\rho_i}\, u_i'(w_{0i})\right)\;.$$

We denote:

$$\Omega \;=\; \times_i\, [\beta_i^{\min}\,,\, \beta_i^{\max}]\,.$$

Also, let

$$\|x\|_{l_\infty} \;=\; \max_i\, |x_i|$$

be the l_∞-norm of a finite sequence, equal to the maximal absolute value of its elements. The following lemma is the main technical result of this section.

Lemma 6.2 (contraction lemma) *For any $C > 0$, the mapping G_C defined via*

$$(G_C)_i(\mathbf{d}) \;=\; \log H_i(C, e^{d_{-i}})$$

[23]We always assume that the price $P_i^{\max}$ is well defined for any investor i. That is, the initial endowment w_{0i} is sufficiently large so that $v_i\,(L_i - e^{-\rho_i}\, E[u_i(w_{1i} + X)])$ is well defined. This assumption is only necessary when dealing with utilities that are defined on a half-line. It can be relaxed at the cost of more technicalities, and we omit it for the reader's convenience.

maps the compact set Ω *into itself and is a strict contraction with respect to* $\|\cdot\|_{l_\infty}$.

Consequently, there exists a unique fixed point $\mathbf{d}^*(C) \in \Omega$ *of this map, solving:*

$$\mathbf{d}^*(C) \;=\; G_C(\mathbf{d}^*(C))\,.$$

For any $\mathbf{d}_0 \in \Omega$*, we have:*

$$\mathbf{d}^*(C) \;=\; \lim_{n\to\infty} (G_C)^n(\mathbf{d}_0).$$

The result of Lemma 6.2 is quite surprising because it holds under absolutely no restrictions on model parameters. In particular, we do not need to impose any smallness conditions that typically are used in economic applications of the contraction mapping theorem. The last technical result we need is the following lemma.

Lemma 6.3 *There exists a unique number* $C^* \in (C_{\min}\,,\, C_{\max})$ *solving:*

$$C \;=\; \left((1-\alpha)\, e^{\rho_S}\, u_S'\left(w_0 \,+\, (1-\alpha)\sum_i \left(w_{0i} \,-\, I_i(e^{d_i^*(C)}\, e^{-\rho_i}\, C^{-1})\right)\right)\right)^{-1}. \qquad (19)$$

Now we are ready to state the main result of this section.

Theorem 6.4 *Let* $\mathbf{b}^*(C^*) \;=\; e^{\mathbf{d}^*(C^*)}$*. The optimal allocation is given by* $(F_i)(\mathbf{b}^*(C^*))$.

Theorem 6.4 provides a directly implementable algorithm for calculating the optimal allocation: The vector $\mathbf{d}(C)$ can be calculated by successive iterations using Lemma 6.2, and then C^* can be found using any standard numerical procedure for solving (19). It is interesting to note that the iterative re-tranching procedure of Theorem 6.4 and Lemma 6.2 has a direct real-world analog. Typically, a CDO issuer re-tranches the underlying pool several times until he finds a tranche structure, which is better suited for the existing investor clientele.

The characterization of the optimal allocation provided by Theorem 6.4 is perfectly suited for studying comparative statics. We will need the following lemma.

Lemma 6.5 (comparative statics lemma) *If the right-hand sides of* (18) *and* (19) *are monotone increasing in some parameter, then so do* C^* *and* $\mathbf{d}^*(C^*)$.

Since, by (12)-(14) and Theorem 5.4, all tranche thresholds and other characteristics of optimal securities can be expressed in terms of $a_i = e^{-d_i}$, we can use Lemma 6.5 to study the dependence of the optimal allocation on various model parameters. Let:

$$Z_{\text{full selling}} \stackrel{def}{=} \max\{X \,:\, F(X) = X\}\,.$$

By Theorem 5.4, we have that $Z_{\text{full selling}} = Z_{J+1}$ is positive if and only if $\max_i Y_i > Y_S$. Also let:

$$\#\{\text{senior}\} = \#\{\mathrm{i} : \mathrm{Y_i} > \mathrm{Y_S}\}$$

be the number of investors participating in the tranches that are fully sold.

If $Z_{\text{full selling}} = 0$, we define:

$$Z_{\text{no trade}} = \max\{x \,:\, F(x) = 0\}$$

to be the threshold Z_N of the super-senior tranche that is not sold at all.

Finally, for each investor i we define:

$$\text{index}(i) = \begin{cases} 1\,, & if \ \text{rank}(i) > J \\ 0\,, & if \ \text{rank}(i) \le J \end{cases}.$$

That is, an investor's index is one if the investor gets a part of the tranches that are fully sold and zero otherwise.

Definition 6.6 *We say that a change in the parameters of the model leads to more selling if it leads to an increase (in the weak sense) in:*

- $\#\{\text{senior}\}$,

- $Z_{\text{full selling}}$,

- $\text{index}(i)$ *for each* i,

and to a decrease (in the weak sense) in $Z_{\text{no trade}}$.

That is, more selling implies that a larger part of X is fully sold and more investors participate in the fully sold senior tranches.

The next proposition describes the effect of a first-order stochastic dominant (FOSD) shift in the distribution of X, as well as the effect of changes in the issuer's initial wealth and discount rate on the optimal allocation.

Proposition 6.7 *A decrease in the distribution of* X *in the FOSD sense, a decrease in* w_0*, and an increase in* ρ_S *leads to more selling. In particular, there exists a threshold value for* ρ_S *such that the super-senior tranche is always sold (retained) by the issuer with* ρ_S *above (below) this threshold,*[24] *and similarly for* w_0.

The intuition behind this result is very clear: A decrease in the FOSD sense makes X less attractive for the issuer and increases her willingness to sell, notwithstanding the fact that the investors are willing to pay less for the tranches. The effects of decreasing w_0 and increasing ρ_S are similar. If an issuer's initial endowment w_0 is small, the marginal value of additional capital increases and forces the issuer to sell more. Similarly, if the rate ρ_S of discounting the cash flows from X is high (equivalently, the firm has other assets with high rates of return available for investment), the issuer's incentive to raise cash gets stronger, which leads to more selling.

We close this section with an analogous comparative statics result for the issuing cost α. It turns out that the situation is more complex. The following is true:

[24] Here, we allow ρ_S to vary and keep the rest of the parameters fixed.

Proposition 6.8 *Suppose that the relative risk aversion* $-\frac{c\,u_S''(c)}{u_S'(c)}$ *of the issuer is above (below)* 1 *for all* c *in the attainable consumption interval* $[\,w_0\,,\,w_0\,+\,(1-\alpha)\,\sum_i\,P_i^{\max}\,]$. *Then, an increase in the cost* α *leads to more (less) selling.*

Thus, the effect of an increase in costs of the optimal allocation depends on the position of the issuer's relative risk aversion with respect to 1. If the issuer is relatively risk tolerant (risk aversion below one), an increase in the costs will force the issuer to retain a larger part of X because trading is not profitable enough. In contrast, if her risk aversion is relatively high (above one), the incentive to sell X will go up, leading to more selling.

7 CARA Investors: Tranching Is Optimal

In this section we consider the benchmark case when all investors, as well as the issuer, have exponential (CARA) utilities[25]:

$$u_i(c) \;=\; A_i^{-1}(1-e^{-A_i c})\;,\; u_S(c) \;=\; A_S^{-1}(1-e^{-A_S c}).$$

In this case, the optimal securities take a particularly simple form. Since for CARA utilities absolute risk tolerance is constant, Proposition 5.5 yields the following:

Proposition 7.1 *For each* i*, the investor* i *gets a portfolio of tranches. Namely, for* $\text{rank}(i) \geq J+1$,

$$F_i \;=\; \sum_{k=0}^{\text{rank}(i)} \kappa_{ik}\,\text{Tranche}_k,$$

[25]This assumption is often used in security design literature. See, for example, Acharya and Bisin (2006).

with

$$\kappa_{ik} \;=\; \begin{cases} \frac{A_i^{-1}}{\sum_{j:\mathrm{rank}(j)\geq k}^{N} A_j^{-1}}\,, & \text{for } k \geq J+1 \\ \frac{A_i^{-1}}{A_S^{-1} + \sum_{j:\mathrm{rank}(j)\geq k+1}^{N} A_j^{-1}}\,, & \text{for } k \;\leq\; J \end{cases}.$$

For $\mathrm{rank}(i) \leq J$,

$$F_i \;=\; \sum_{k=0}^{\mathrm{rank}(i)-1} \kappa_{i\,k}\,\mathrm{Tranche}_k,$$

with

$$\kappa_{i\,k} \;=\; \frac{A_i^{-1}}{A_S^{-1} \,+\, \sum_{j:\mathrm{rank}(j)\geq k+1} A_j^{-1}}.$$

The result of Proposition 7.1 is very important in view of its relation to the securitization practice observed in the financial industry. Namely, if X is a pool of morgages, bonds, or commercial loans held by a bank, the bank often generates a CDO by forming multiple tranches from the pool of the assets and sells portfolios of such tranches to investors. Proposition 7.1 shows that this often-observed securitization practice has a rational explanation if we adopt the hypothesis that economic agents participating in the CDO markets have CARA preferences. This is not an unnatural assumption. For example, Gollier (2004) notes that the behavior of economic agents is consistent with CARA preferences when the wealth level is high. Thus, CARA assumption should work well if the wealth of the agents participating in the CDO market is high relative to the size of the pool X.

We now discuss the structure of real-world CDOs in greater detail. There are two types of CDO transactions: true-sale transactions and synthetic transactions. In a true-sale transaction, the underlying credit exposures are taken on using physical assets (e.g., bonds or mortgages.) The originator of a true sale transaction always fully sells the senior tranche ($\mathrm{Tranche}(0,b)$ for some $b > 0$ in our notation) and never fully sells the junior tranche. The quantity

$$\bar{X} \,-\, b$$

is usually referred to as the first loss position (FLP). It absorbs all default losses up to a limit, equal to its volume $\bar{X} - b$. The commonly accepted intuition behind this behavior is that, in the presence of asymmetric information, the most junior tranche is the most information-sensitive. Therefore, the issuer retains this tranche (or a part of it) to reduce investor skepticism, driven by information asymmetries. Proposition 7.1 offers an alternative explanation for this phenomenon, based on the investors' risk aversion.

For a synthetic CDO transaction, the underlying credit exposures are taken on using a credit default swap rather than by having a vehicle buy physical assets. A synthetic CDO is always characterized by the presence of the so-called second loss position (SLP) and the third loss position (TLP). Namely, for a given single-tranche CDO (Tranche(a, b) is our notation[26]), the SLP

$$\bar{X} - a$$

ensures that the investor bears only part (equal to $b - a$) of the default losses beyond the FLP. The most puzzling property of all synthetic CDO transactions is the presence of the third loss position (TLP). Namely, the issuer always retains the super-senior Tranche$(0, c)$, in which case c is the TLP. This phenomenon stands in stark contrast to the predictions of most existing models of optimal security design based on asymmetric information (see, e.g., DeMarzo and Duffie (1999), DeMarzo (2005)) where it is optimal to sell the super-senior tranche because it is the least information-sensitive.

Theorem 5.4 and Proposition 7.1 provide a natural theoretical explanation of this phenomenon: When the issuer is risk averse and the MMRIS of all investors are small relative to that of the issuer, the issuer will always retain the super-senior tranche. Furthermore, our model generates a qualitative behavior of the tranche thresholds, consistent with that observed in the data. We appeal to Franke, Herrmann, and Weber (2007), where they study numerous relations between the properties of the underlying

[26]In a real-world transaction, this tranche may consist of multiple component tranches.

pool, the properties of the issuer, and the structure of CDO transactions. Here, we formulate and provide theoretical justification for three of them:

(1) Transactions with a TLP are preferred to true-sale transactions for asset pools with high quality;

(2) In a transaction with a TLP, the size of the non-securitized super-senior tranche (TLP) increases with the quality of the asset pool; and

(3) Transactions with a TLP (without a TLP) are preferably used by banks with a strong (weak) rating.

To study (1) and (2) in our stylized theoretical setup, we say that a pool X_1 is of higher quality than X_2 if the distribution of X_1 obtains as a FOSD shift in the distribution of X_2. By Proposition 6.7, a sufficiently strong increase in the pool quality induces the issuer to retain the super-senior tranche, and she will prefer a synthetic transaction in agreement with the stylized fact (1). Furthermore, by Proposition 6.7, the TLP$= Z_{\text{no trade}}$ is monotone increasing in the quality of the pool, in complete agreement with (2).

A bank with a weak rating faces a higher interest rate on borrowing. Therefore, the discount rate ρ_S for such a bank will be higher, driven by the stronger incentive to raise capital. Similarly, a bank also has a weak rating if its initial capital w_0 is small. Stylized fact (3) follows directly from Proposition 6.7. We summarize these findings in the following proposition.

Proposition 7.2 *Stylized facts (1)-(3) hold true.*

Note that we could also use the comparative statics results of Proposition 6.8 to make testable empirical predictions. With CARA utilities, relative risk aversion of the issuer is given by $c\, A_S$. If we adopt the hypothesis that the issuer's capital is typically large and absolute risk aversion A_S is not too small, we should observe a smaller TLP in markets with larger proportional issuing costs.

Another interesting finding of Franke, Herrmann, and Weber (2009) is that the FLP decreases with the quality of the asset pool.[27] When the issuer is risk averse, the FLP is given by $\bar{X} - Z_1$. That is, Z_1 must be monotone increasing in the pool quality. Even though we have not proven this finding theoretically, numerical simulations suggest that it is often true.

We now use the profiles in Table 1 to run numerical computation.

We first consider the case when X either takes value 1 with probability $1-\delta$ or a random value, uniformly distributed on $[0,1]$ with probability δ. If X is a pool of bonds with total face value 1, then δ can be naturally interpreted as the default probability. Figure 1(a) shows a plot of the tranche thresholds $Z_1, \cdots, Z_5$ against the default probability δ.

Then, we fix the default probability $\delta = 0.5$ and vary the distribution of the recovery rate. Conditioned on the default event, we allow X to be distributed with the density $n\, x^{n-1}$ on $[0,1]$ for some $n > 0$. Clearly, increasing n makes the distribution of X positively skewed and improves the pool quality in the FOSD sense. Figure 1(b) shows a plot of the tranche thresholds $Z_1, \cdots, Z_5$ against the skewness parameter n.

To illustrate the effect of ρ_S on the optimal allocation, we fix $\delta = 0.5$ and assume as above that X is uniformly distributed on $[0,1]$. Figure 2 shows a plot of the tranche thresholds $Z_1, \cdots, Z_5$ against the discount rate ρ_S of the issuer. Figures 1(a), 1(b) and 2 clearly show that the TLP (threshold Z_5 in our setting) is monotone increasing in the quality of the pool and decreasing in the issuer's discount rate ρ_S while the FLP ($\bar{X} - Z_1$ in our setting) is monotone in the opposite directions with respect to these parameters,

[27] Other empirical findings of Franke, Herrmann, and Weber (2009) are related to information asymmetries and cannot be tested in our model.

Table 1: Agent Profiles with Risk-Averse Issuer

Agent	ρ	A	w_0	w_1
Issuer	0.08	1	0	0
Investor 1	0.01	0.3	0	0
Investor 2	0.04	0.4	0	0
Investor 3	0.06	0.6	0	0
Investor 4	0.08	0.1	0	0
Investor 5	0.1	0.1	0	0

in complete agreement with the above mentioned stylized facts.

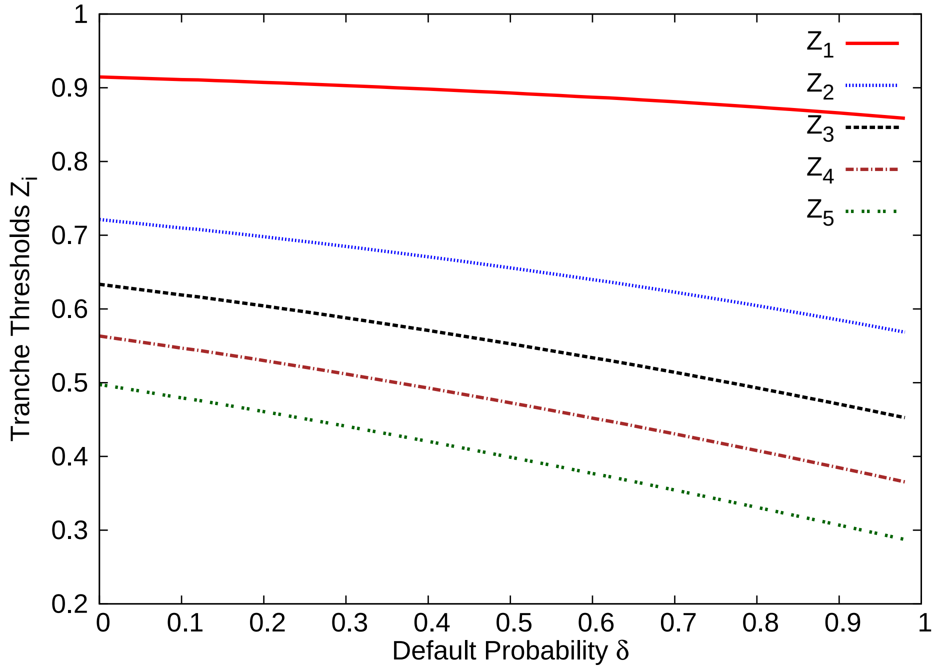
We complete this section with two important results that hold only when investors have CARA preferences.

In general, the ranks that the issuer assigns to the investors may depend in a non-trivial way on investors' preferences and endowments. However, it turns out that, when all investors have CARA preferences, ranks can be characterized explicitly. The following is true:

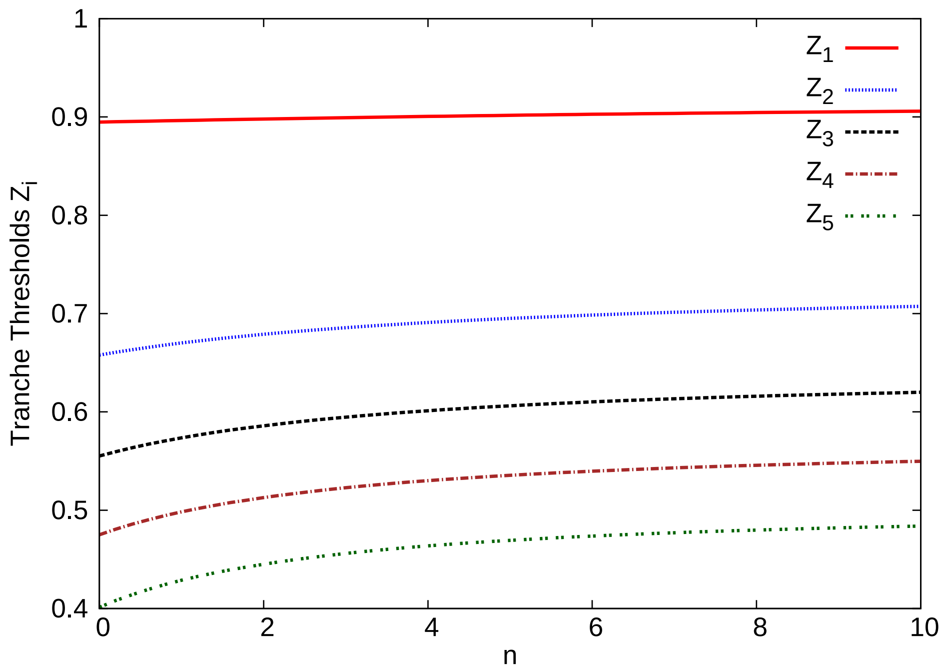
Proposition 7.3 *The ranks of the investors follow the order of their pre-trade MRIS. That is* $\mathrm{rank}(i) > \mathrm{rank}(j)$ *if and only if*

$$\frac{e^{-\rho_i}\, u_i'(w_{1i})}{u_i'(w_{0i})} \;>\; \frac{e^{-\rho_j}\, u_j'(w_{1j})}{u_j'(w_{0j})}.$$

Suppose for simplicity that the investors' endowments satisfy $w_{0i} = w_{1i}$. In this case, Proposition 7.3 implies that the rank of an investor is determined solely by his discount rate ρ_i and is independent of his risk aversion A_i. The reason is that, when an investor's risk aversion is constant, the risk premium *per unit of risk* that the investor is charging for holding a risky cash flow is independent of the level of X. By Proposition 7.1, for any investor i, the issuer optimally chooses the fraction of the total cash flow $F(X)$ that investor i gets to be proportional to his risk tolerance A_i, thereby equalizing marginal



(a) Change Distribution Through Default Probability



(b) Change Distribution Through Skewness

Figure 1: The Effect of Asset Pool Quality

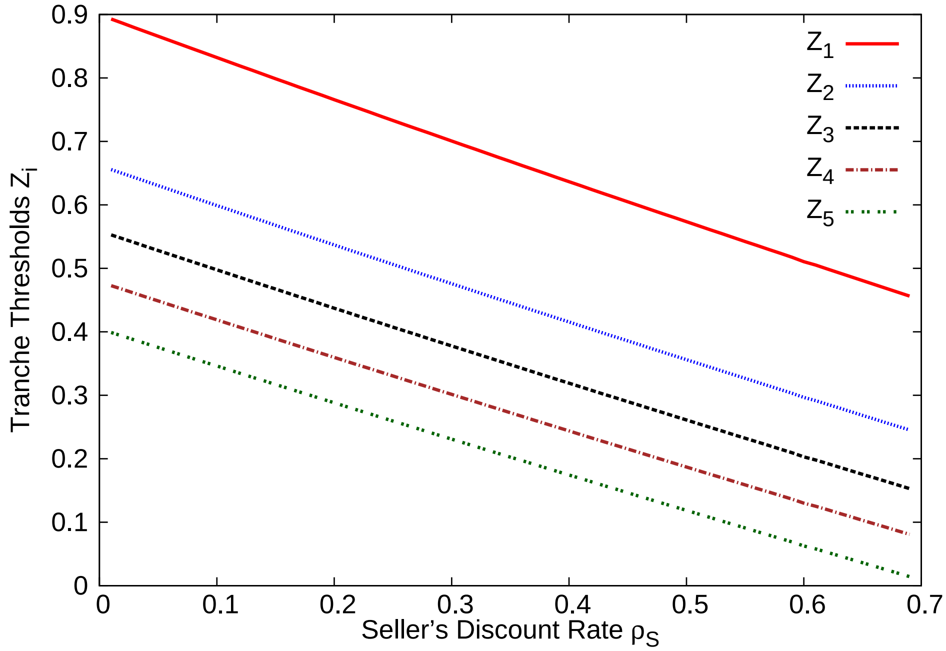


Figure 2: The Effect of Issuer's Discount Rate

risk premia across the investors. Therefore, only discount rates ρ_i matter for ranking. In particular, if several investors have identical pre-trade MRIS, the tranches they will participate in will be the same, and the portfolios of tranches they get will be identical, up to a constant multiple. This leads to the following interesting aggregation result. Denote by A_B^{-1} the sum of investors' risk tolerances:

$$A_B^{-1} \;=\; \sum_{j=1}^{N} A_j^{-1}. \tag{20}$$

The following is true:

Proposition 7.4 *Suppose that* $w_{0i} \;=\; w_{1i}$[28] *for all* i*, and* $\rho_i = \rho_B$ *is independent of* i*.*

[28]Due to translation invariance of CARA preferences, optimal allocation depends only on the differences $w_{1i} - w_{0i}$ and $w_1 - w_0$ of endowments at times zero and one.

Then, the risk-sharing is linear:

$$F_i(x) \;=\; \frac{A_i^{-1}}{A_B^{-1}}\, F(x)$$

and $F(x)$ *coincides with the optimal security that the issuer would sell to a single representative investor with risk aversion* A_B.

There exist constants:

$$K_{\max} \;>\; K_{\mathrm{mid}} \;>\; K_{\min}$$

such that the following is true:

- *If*

  $$\rho_S - \rho_B \;>\; K_{\max},$$

  then full selling is optimal, $F(X) = X$;

- *If*

  $$K_{\max} \;>\; \rho_S - \rho_B \;>\; K_{\mathrm{mid}},$$

  then $F(X)$ *is a combination of a standard debt and a fraction of the junior equity tranche,*

  $$F(X) \;=\; F(X,a) \;=\; \min(X\,,\, Z(a)) \;+\; \frac{A_B^{-1}}{A_S^{-1} + A_B^{-1}}\, \max(X - Z(a)\,,\, 0). \qquad (21)$$

- *If*

  $$K_{\mathrm{mid}} \;>\; \rho_S - \rho_B \;>\; K_{\min},$$

  then the issuer retains the senior tranche and sells a fraction of the junior equity tranche,

  $$F(X) \;=\; F(X,a) \;=\; \frac{A_B^{-1}}{A_S^{-1} + A_B^{-1}}\, \max(X - Z(a)\,,\, 0). \qquad (22)$$

- *If*

$$K_{\min} > \rho_S - \rho_B ,$$

then there is no trade, $F(X) = F_i(X) = 0$ *for all* i.

Furthermore,

$$Z(a) \;=\; \begin{cases} A_B^{-1}(\log a + A_S w_1) & \textit{in case } (2) \\ -A_S^{-1}\,\log a & \textit{in case } (3) \end{cases},$$

and a *is the unique solution to*

$$a \;=\; e^{\rho_S-\rho_B}\, e^{-A_S w_0}\, \left(1 + e^{-\rho_B}\, E[1 - e^{-A_B\, F(X,a)}]\right)^{-(A_B+A_S)/A_B}\;. \tag{23}$$

Proposition 7.4 shows that, when all investors have CARA utilities with identical endowments and discount rates, the allocation is characterized by the presence of a representative investor such that the total optimal security $F(x)$ coincides with that in an artificial market, populated by this single representative investor. This phenomenon is known as aggregation and also arises in the Sharpe-Lintner capital asset pricing model (CAPM) (see, e.g., Sharpe (1964)) as well as in general complete market Walrasian equilibrium allocations. Indeed, by the First Welfare Theorem, such allocations are always Pareto-efficient and can therefore be characterized by the presence of a single representative investor. In contrast, in our model, aggregation only takes place under very special conditions, such as those of Proposition 7.4.

8 Risk-Neutral Issuer

Risk neutrality of the issuer is a common assumption in the literature on security design. It is usually made in order to focus on liquidity motives as opposed to risk-sharing motives for securitization. In fact, many motivating examples involve the sale of securities by previously incorporated and publicly traded firms. For such cases, the firm does not come directly equipped with an attitude toward risk. With liquid capital markets, for example, Modigliani and Miller (1958) and Stiglitz (1974) showed that a publicly traded firm is indifferent to certain kinds of financial risk. In this section we present a detailed study of optimal securitization when the issuer is risk neutral.

We start with the following proposition:

Proposition 8.1 *If the issuer is risk neutral, then the full optimal security $F(X)$ is a standard debt:*

$$F(X) \;=\; \text{Tranche}(0, Z_{J+1}).$$

Furthermore, there is trade (i.e., $Z_{J+1} > 0$) if and only if:

$$e^{-\rho_S} \;\leq\; \max_i \frac{e^{-\rho_i}\, u_i'(w_{1i})}{u_i'(w_{0i})}.$$

By Proposition 5.5, the slope of $F_i(X)$ is equal to zero for $X \geq Z_{J+1}$ when the issuer is risk neutral because her risk tolerance is infinite and so she will always either fully sell all the risks or fully bear them and will never participate in a non-trivial risk sharing for a given level of X.

Furthermore, the independence of the MRIS of a risk-neutral issuer of her consumption makes it possible to study the dependence of optimal allocation on the characteristics of the investors. Since, by Lemma 6.3, the factor C in this case equals $e^{-\rho_S}$, a change in the characteristics of the investors does not have any effect on C. Therefore, to apply Lemma 6.5, it suffices to establish the required monotonicity in (18). The following is true:

Proposition 8.2 *If the issuer is risk neutral, then adding an investor to the population of investors and/or increasing the initial endowment w_{0i} of some investor i always leads*

Table 2: Agent Profiles with Risk-Neutral Issuer

Agent	ρ	A	w_0	w_1
Issuer	0.08	0	0	0
Investor 1	0.01	0.1	0	0
Investor 2	0.03	0.3	0	0
Investor 3	0.04	0.2	0	0
Investor 4	0.05	0.5	0	0
Investor 5	0.065	0.4	0	0

to more selling.[29]

Clearly, introducing a new investor has a positive effect for the issuer: When there are more investors, the issuer's opportunities are better. In particular, the issuer optimally increases $Z_{\text{full selling}} = Z_{J+1}$ in response to the presence of a new investor. Similarly, an increase in the investor's initial endowment increases his willingness to buy in order to reallocate his wealth between time periods.

We use the profile in Table 2 to illustrate the result numerically. Figure 3 and 4 show the optimal security structure as we increase the number of investors from 2 to 5.

We complete this section with a very intuitive result, illustrating the effect of changes in investors' discount rates on optimal allocation. To state the result, we need the following definition:

Definition 8.3 *We say that the size of X is mild for investor i if*

$$(u_i(w_{0i}) - u_i(c)) \frac{1}{R_i(c)} \leq u_i'(c) \tag{24}$$

for all c in the attainable consumption interval,

$$c \in [w_{0i} - P_i^{\max}, w_{0i}].$$

[29]See Definition 6.6.

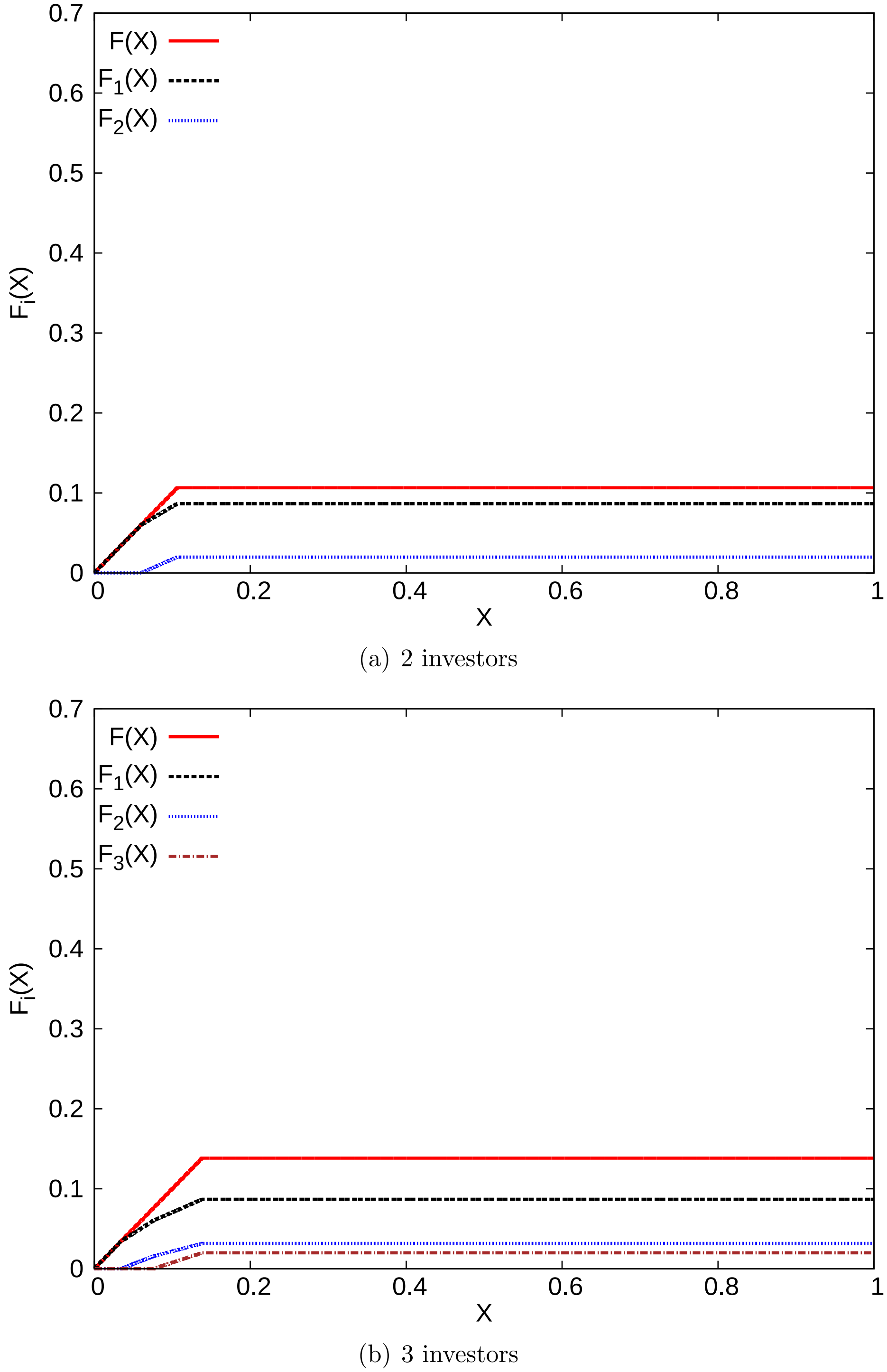
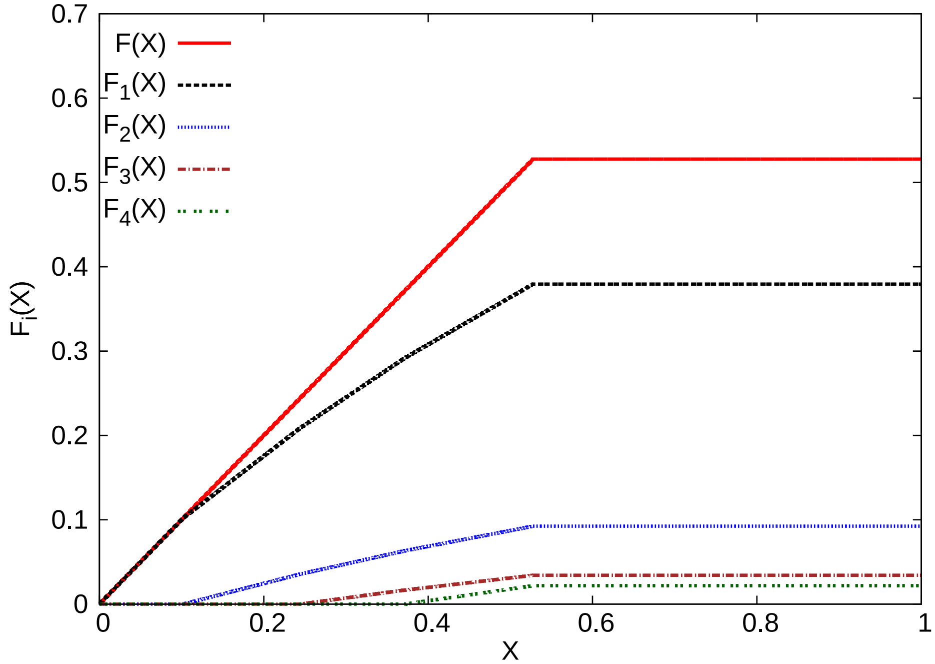
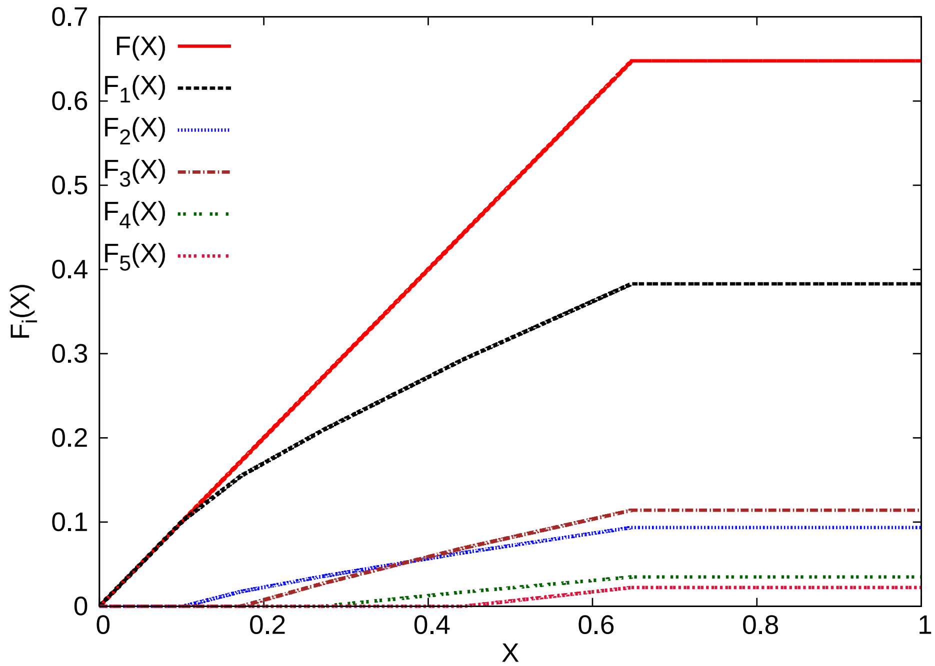


Figure 3: The Effect of Adding More Investors (2 investors and 3 investors)



(a) 4 investors



(b) 5 investors

Figure 4: The Effect of Adding More Investors (4 investors and 5 investors)

Condition (24) has a clear economic meaning: Whatever investor i buys, his total utility loss, normalized by the risk tolerance, will be smaller than the marginal loss $u_i'(c)$. Clearly, this condition requires that the size of X not be too large relative to the investor's wealth. The following is true:

Proposition 8.4 *Suppose that the issuer is risk neutral and that the size of X is mild for investor i. Then, a decrease in ρ_i leads to more selling.*

The intuition behind Proposition 8.4 is completely analogous to that described above: If the discount rate of an investor i decreases, his valuation of future cash flows increases. This creates better opportunities for the issuer and induces her to sell more. The requirement that the size of X be mild is not too restrictive. For example, the following is true:[30]

Lemma 8.5 *If investor i has a CARA utility, then the size of X is always mild for him.*

If investor i has a constant relative risk aversion utility $u_i(c) = (c^{1-\gamma} - 1)/(1-\gamma)$, then the size of X is mild for him if:

$$\begin{cases} e^{-\rho_i}\, E\left[\left(\frac{w_{1i}+X}{w_{0i}}\right)^{1-\gamma} - \left(\frac{w_{1i}}{w_{0i}}\right)^{1-\gamma}\right] \;\le\; 1-\gamma\,, & \gamma < 1 \\ e^{-\rho_i}\, E\left[\log\left(\frac{w_{1i}+X}{w_{1i}}\right)\right] \;\le\; 1\,, & \gamma = 1 \\ e^{-\rho_i}\, E\left[\left(\frac{w_{1i}}{w_{0i}}\right)^{1-\gamma} - \left(\frac{w_{1i}+X}{w_{0i}}\right)^{1-\gamma}\right] \;\le\; \gamma - 1\,, & \gamma > 1 \end{cases}$$

In particular, the size of X is always mild for him if either w_{1i} or his discount rate ρ_i is sufficiently large.

Finally, we note that the results of Propositions 8.2 and 8.4 do not generally hold when the issuer is risk averse. The reason is that, when the issuer is risk averse, the

[30] The proof follows by direct calculation.

marginal value of an additional unit of consumption at time zero decreases when there are more opportunities to raise cash. This effect may overcome the incentive to sell a larger part of X and drive selling down. We again use the profiles in Table 1 to numerically illustrate this point in Figure 5. As we can see from the plot, the decrease of the first investor's discount rate does not necessarily lead to more selling because the super-senior tranche is not sold and $Z_{notrade}$ decreases as ρ_1 increases over the interval of $[0.04, 0.05]$.

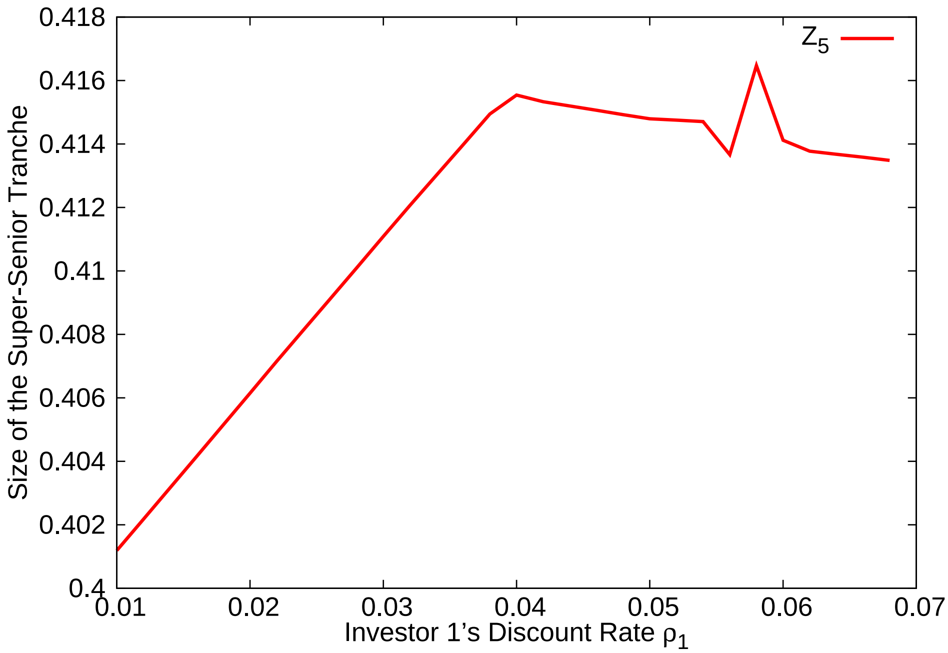


Figure 5: A Counter Example

9 Fixed Costs of Issuing Securities

In this section we discuss the effects of fixed securitization costs on the optimal allocation. Namely, we assume that issuing any security costs a fixed amount C [31] in addition to the

[31] This is also the case studied by Allen and Gale (1989).

already included proportional costs. In this case, the optimal securitization problem of the issuer contains a discrete component. The issuer has to decide how many securities she will issue and which investors she will sell them to. Then, given the chosen investors and the number K of the securities the issuer decides to issue, the problem reduces to the one studied in the previous sections, but with the initial endowment of the issuer given by $w_0 - KC$. We summarize the basic implications of costs in the following proposition:

Proposition 9.1 *We have:*

(1) The optimal number $K = K(C)$ of issued securities is monotone decreasing in cost C;

(2) There exist thresholds

$$0 = C_{N+1} \le C_N \le C_{N-1} \le \cdots \le C_1 \le C_0 = 0,$$

such that it is optimal to issue K securities if $C \in (C_{K+1}, C_K)$;

(3) If the issuer is risk neutral and a decrease in the cost does not force the issuer to drop any investors, then this decrease in cost C leads to more selling;

(4) If the issuer is risk averse, then for $C \in (C_{K+1}, C_K)$, an increase in C leads to more selling if this increase does not lead to changing investors.

Items (1) and (2) follow directly from the definition of the problem: The higher is the cost of issuing securities, the less is the incentive for the issuer to issue. To explain the meaning of items (3) and (4), we note that a decrease in the cost of issuing may force the issuer to choose another group of investors rather than adding new investors to already existing ones. For example, suppose that the cost is so high that the issuer only sells a security to investor 1. If the cost decreases, it may be optimal for the investor to trade only with investors 2 and 3 rather than with 1 and 2 or 1 and 3. In this case, investor 1 is dropped. But if nobody gets dropped, a decrease in the cost simply induces the issuer to add new investors to already existing ones, which, by Proposition 8.2, always leads to more selling and (3) follows. If the issuer is risk averse, an increase in the cost reduces her initial consumption and therefore increases her incentive to raise capital. As long as the set of investors to which she is selling does not change, (4) immediately follows from Proposition 6.7.

Table 3 shows how the optimal number of investors depends on the cost C. The profiles of the issuer and the investors are given in Table 1 and the proportional cost level is set to be 5%.

Table 3: Optimal Selection of Investors with Fixed and Proportional Cost

C in %	0.01	0.1	0.5	2	13
Investors	1,2,3,4	1,2,4	1,4	4	–
Issuer's Expected Utility	0.599129	0.596688	0.589469	0.57357	0.49502

We see here that a surprising phenomenon occurs: Even though investor 4 has the highest discount rate and therefore the lowest rank among the first four investors (by Proposition 7.3), the issuer always sells a security to him because of his very low risk aversion. In contrast, even though investors $1, 2, 3$ have lower discount rates, they get dropped when the cost C increases because of their high risk aversion. This phenomenon illustrates that, when the issuing costs are high, the risk premia that the investors are charging play a crucial role in the issuer's decision on which investors to choose.

10 Conclusions

We solve the problem of optimal securitization for an issuer facing multiple investors with arbitrary heterogeneous risk attitudes, discount rates, and endowments, and without asymmetric information. We show that optimal securities can be characterized as portfolios of multiple non-linear tranches and have a prioritized structure: The issuer optimally assigns ranks to investors depending on their MMRIS, and based on these ranks, the issuer determines the optimal tranche thresholds. She then sells either the super-senior or the second senior tranche to the investor of highest rank, and then gradually sells the mezzanine and junior tranches to investors with lower ranks, so that every subsequent tranche is shared by multiple investors in a Pareto-efficient way. In particular, the junior (equity) tranche is never fully sold. [32]

When the issuer and all investors have exponential (CARA) utilities, optimal securities are given by portfolios of standard linear tranches (CDOs). In particular, when the issuer is risk averse and her discount rate is not too large, she optimally retains the super-senior tranche and sells portfolios of fractions of senior and junior tranches to the investors. The model generates theoretical predictions about the dependence of the non-securitized super-senior tranche (TLP) on underlying microeconomic characteristics that are confirmed by recent empirical findings.

We conclude that risk-sharing motives and the risk aversion of both the issuer and the investors are driving forces for securitization that are at least as important as asymmetric information. To the best of our knowledge, this is the first model in the literature that explains the appearance of multiple tranches in the CDO design and the relation of the tranche thresholds to microeconomic characteristics. We believe that these results are of both theoretical and practical importance and can be used by banks and intermediaries to improve securitization.

Our model and algorithm could also be used to compute CDO-squared. For example, the original investor himself could have specific knowledge about potential investors he represents (their risk attitudes and discount rates). He could then re-tranche the

[32] Unless the junior (equity) tranche collaps to a single point and the MMRIS of all investors are larger than that of the issuer.

portfolio that he has purchased from the original issuer and sell to his clients. Another example would be the case when the issuer re-tranches an unsold tranche and sells to new investors.

Finally, we note that the model could also be viewed as surplus extraction through price discrimination when the issuer has complete information. It would be interesting to extend our model to the case when the issuer has incomplete information about investor types, as in the model of Cremer and McLean (1985). It would also be interesting to extend our model to a dynamic, multi-period setting and allow for asymmetric information. Our techniques for analyzing constrained efficient allocations could also be applied to models outside of security design, such as, for example, equilibrium models with participation constraints. See Alvarez and Jermann (2000,2001). We leave these for future research.

Appendix

A One Investor

Proof of Theorem 4.1. The first-order Kuhn-Tucker conditions are

$$\begin{aligned} & - e^{-\rho_S}\, u_S'(w_1 + X - F(X)) \\ & + (1-\alpha)\, u_S'(w_0 + (1-\alpha)\, P)\, v_B' \left(L_B - e^{-\rho_B}\, E[u_B(w_{1B} + F)]\right)\, e^{-\rho_B}\, u_B'(w_{1B} + F) \;=\; 0 \end{aligned} \tag{25}$$

if the constraints $0 \leq F(X) \leq X$ are not binding; the form

$$
\begin{aligned}
& - e^{-\rho_S} u_S'(w_1 + X - F(X)) \\
& \quad + (1-\alpha)\, u_S'(w_0 + P)\, v_B' \left(L_B - e^{-\rho_B} E[u_B(w_{1B} + F)]\right) e^{-\rho_B} u_B'(w_{1B} + F) > 0
\end{aligned}
\tag{26}
$$

with $F(X) = X$ if the constraint $F(X) = X$ is binding; and the form

$$
\begin{aligned}
& - e^{-\rho_S} u_S'(w_1 + X - F(X)) \\
& \quad + (1-\alpha)\, u_S'(w_0 + P)\, v_B' \left(L_B - e^{-\rho_B} E[u_B(w_{1B} + F)]\right) e^{-\rho_B} u_B'(w_{1B} + F) < 0
\end{aligned}
\tag{27}
$$

with $F(X) = 0$ if the constraint $F(X) = 0$ is binding.

Since the maximization problem for the issuer is strictly concave, to prove the theorem it suffices to check that the first-order conditions, (25) through (27), hold true.[33] We consider all four cases identified in the theorem.

(1) and (4). We only prove (1). Case (4) is analogous. To show that $F(X) = X$, we need to show that the constraint is binding. That is, (26) holds with $F(X) = X$. Using the identity

$$
v_B'(x) = 1/u_B'(v_B(x)), \tag{28}
$$

we get

$$
-e^{-\rho_S} u_S'(w_1) + (1-\alpha) \frac{u_S'(w_0 + (1-\alpha) P_{\max})}{u_B'(w_{0B} - P_{\max})} e^{-\rho_B} u_B'(w_{1B} + X) > 0
$$

[33]Even though the problem is infinite dimensional, sufficiency of the Kuhn-Tucker conditions can be verified directly by standard methods due to the strict concavity of the problem. See, e.g., Seierstad and Sydsaeter (1977) and Mitter (2008).

for all $X \in [0, \bar{X}]$. Since u'_B is decreasing, it suffices to check it for $X = \bar{X}$. This is equivalent to the condition $\rho_S - \rho_B > K_{\max}$.

(2) and (3). For simplicity, we only prove (3). Case (2) is analogous. First, we need to show that the equation for a does have a solution. To this end, by the Brower fixed-point theorem, it suffices to show that the right-hand side of (6) maps a compact interval into itself. This is clear because:

$$u_B(w_{1B}) \ \le \ E[u_B(w_{1B} + F_a(X))] \ \le \ E[u_B(w_{1B} + X)]\,.$$

Now, we claim that the right-hand side of (6) is monotone decreasing in a. Indeed, to prove it, it suffices to show that

$$E[u_B(w_{1B} + F_a(X))]$$

is monotone increasing in a, since the numerator is decreasing in $E[u_B(w_{1B} + F_a(X))]$ and the denominator is decreasing in it. Differentiating (4) with respect to a, we get:

$$\frac{\partial g}{\partial a} \ = \ -\frac{u'_B(w_{1B} + g)}{a\, u''_B(w_{1B} + g) + u''_S(w_1 + x - g)}\,,$$

and therefore g is monotone increasing in a. Also, $g(a, Z(a), w_{1B}) = 0$. Therefore,

$$
\begin{aligned}
&\frac{\partial}{\partial a}\left(E[u_B(w_{1B} + F_a(X))]\right) \\
&= \frac{\partial}{\partial a}\left(u_B(w_{1B}) \int_0^{Z(a)} p(x)dx \;+\; \int_{Z(a)}^{\bar{X}} u_B(w_{1B} + g(a, X))\, p(x)\, dx\right) \\
&= u_B(w_{1B}) \frac{\partial}{\partial a} \int_0^{Z(a)} p(x)dx \;-\; u_B(w_{1B}) \frac{\partial}{\partial a} \int_0^{Z(a)} p(x)dx \qquad (29) \\
&+ \int_{Z(a)}^{\bar{X}} u_B'(w_{1B} + g(a, x))\, \frac{\partial g}{\partial a}(a, x)\, p(x)\, dx \\
&= \int_{Z(a)}^{\bar{X}} u_B'(w_{1B} + g(a, x))\, \frac{\partial g}{\partial a}(a, x)\, p(x)\, dx \;>\; 0,
\end{aligned}
$$

and the claim follows. ■

B Kuhn-Tucker First-Order Conditions for Multiple Investors

By strict concavity, an allocation is optimal if and only if it satisfies the first-order Kuhn-Tucker conditions. They are:

$$
\begin{aligned}
&- e^{-\rho_S}\, u_S'(w_1 + X - F(X)) \\
&+ (1-\alpha)\, u_S'\left(w_0 + (1-\alpha) \sum_i P_i\right) v_i'\left(L_i \;-\; e^{-\rho_i}\, E[u_i(w_{1i} + F_i)]\right) e^{-\rho_i}\, u_i'(w_{1i} + F_i) \;=\; 0
\end{aligned} \qquad (30)
$$

if the constraints $F_i \geq 0$ and $\sum_j F_j \;\leq\; X$ are not binding, and

$$
\begin{aligned}
&- e^{-\rho_S}\, u_S'(w_1 + X - F(X)) \\
&+ (1-\alpha)\, u_S'\left(w_0 + (1-\alpha) \sum_i P_i\right) v_i'\left(L_i \;-\; e^{-\rho_i}\, E[u_i(w_{1i} + F_i)]\right) e^{-\rho_i}\, u_i'(w_{1i} + F_i) \;<\; 0
\end{aligned} \qquad (31)
$$

if the constraint $F_i \geq 0$ is binding but the constraint $\sum_j F_j \leq X$ is not binding.

Finally, if the constraint $\sum_j F_j \leq X$ is binding, there will be a Lagrange multiplier $\nu(X)$ for this constraint, and the first-order condition will be

$$\begin{aligned} & - e^{-\rho_S}\, u_S'(w_1 + X - F(X)) \\ & + \ (1-\alpha)\, u_S' \left(w_0 + (1-\alpha) \sum_i P_i \right) v_i' \left(L_i \ - \ e^{-\rho_i}\, E[u_i(w_{1i} + F_i)] \right) e^{-\rho_i}\, u_i'(w_{1i} \ + \ F_i) \\ & \qquad = \ \nu(X) \ > \ 0 \end{aligned} \tag{32}$$

if the constraint $F_i \geq 0$ is not binding, and

$$\begin{aligned} & - e^{-\rho_S}\, u_S'(w_1 + X - F(X)) \\ & + \ (1-\alpha)\, u_S' \left(w_0 + (1-\alpha) \sum_i P_i \right) v_i' \left(L_i \ - \ e^{-\rho_i}\, E[u_i(w_{1i} + F_i)] \right) e^{-\rho_i}\, u_i'(w_{1i} \ + \ F_i) \\ & \qquad < \ \nu(X) \end{aligned} \tag{33}$$

if the constraint $F_i \geq 0$ is binding. By (28),

$$v_i' \left(L_i \ - \ e^{-\rho_i}\, E[u_i(w_{1i} + F_i)] \right) \ = \ \frac{1}{u_i'(w_{0i} - P_i)} \ = \ \frac{1}{u_i'(c_{0i})}\,,$$

and therefore, by (11),

$$\begin{aligned} a_i \ &= \ e^{\rho_S - \rho_i}\, \frac{(1-\alpha)\, u_S' \left(w_0 + (1-\alpha) \sum_i P_i \right)}{u_i' \left(v_i \left(L_i \ - \ e^{-\rho_i}\, E[u_i(w_{1i} + F_i)] \right) \right)} \\ &= \ e^{\rho_S - \rho_i}\, (1-\alpha)\, u_S' \left(w_0 + (1-\alpha) \sum_i P_i \right) v_i' \left(L_i \ - \ e^{-\rho_i}\, E[u_i(w_{1i} + F_i)] \right) . \end{aligned} \tag{34}$$

Thus, we can rewrite (30) through (33) in the form:

$$a_i\, u_i'(w_{1i} \, + \, F_i(X)) \; = \; u_S'(w_1 + X - F(X)) \tag{35}$$

when none of the constraints is binding, and

$$a_i\, u_i'(w_{1i} \, + \, F_i(X)) \; < \; u_S'(w_1 + X - F(X)) \tag{36}$$

when the constraint $F_i \geq 0$ is binding but the constraint $\sum_j F_j \leq X$ is not.

When the constraint $\sum_j F_j(X) \leq X$ is binding, we have $\sum_j F_j(X) = X$. If we set:

$$\lambda(X) \; \stackrel{def}{=} \; e^{\rho S}\, \nu(X) \, + \, u_S'(w_1),$$

then, (32) and (33) take the form

$$a_i\, u_i'(w_{1i} \, + \, F_i(X)) \; = \; \lambda(X) \; > \; u_S'(w_1) \tag{37}$$

when $F_i \; \geq \; 0$ is not binding and

$$a_i\, u_i'(w_{1i} \, + \, F_i(X)) \; < \; \lambda(X) \tag{38}$$

when it is binding.

By abuse of notation, we from now on reorder the investors in the increasing order of their rank. In other words, investor i means from now on the investor whose rank is equal to i.

By the uniqueness of optimal allocation, it suffices to show that the allocation, described in Proposition 5.4 and Theorem 5.5, indeed satisfies the first-order conditions (35) through (38). This is done in subsequent lemmas.

Lemma B.1 *Let $k \geq J+1$. Then, for all $X \in [Z_{k+1}, Z_k]\,(= \text{Tranche}_k)$, the constraint $\sum_j F_j(X) \leq X$ is binding and the constraint $F_j(x) \geq 0$ is binding for all $j < k$. The optimal allocation for $X \in \text{Tranche}_k$ is uniquely determined via*

$$F_j(X) \;=\; \begin{cases} I_j(\lambda_k(X)\, a_j^{-1}) \;-\; w_{1j}\;, & j \geq k \\ 0\;, & j < k \end{cases}. \tag{39}$$

Here, $\lambda_k(X)$ is the unique solution to

$$X \;=\; \sum_{j\geq k} \left(I_j(\lambda_k(X)\, a_j^{-1}) \;-\; w_{1j}\right). \tag{40}$$

The slope of $F_j(X),\ j \geq k$ satisfies

$$\frac{d}{dx} F_j(X) \;=\; \frac{R_j(c_{1j})}{\sum_{i\geq k} R_i(c_{1i})}.$$

Proof. By construction, the conjectured optimal allocation satisfies

$$\sum_j F_j(X) \;=\; X$$

for all $X \leq Z_{J+1}$. Thus, we need to verify that (37) and (38) hold in this case. Here, the connection between $\mu_k(X)$ from Proposition 5.4 and $\lambda_k(X)$ is given by:

$$\mu_k(X) \;=\; \frac{\lambda_k(X)\, e^{-\rho S}}{u_S'(c_{0S})}.$$

By (39) and (40), $F_i(X)$ satisfies

$$a_i\, u_i'(w_{1i} \,+\, F_i(X)) \;=\; \lambda_k(X) \qquad \text{and} \qquad \sum_i F_i(X) \;=\; X,$$

and it remains to check that equation (40) has a solution $\lambda_k(X)$ such that

$$\lambda_k(X) \;\geq\; u'_S(w_1) \tag{41}$$

(constraint $\sum_j F_j \leq X$ is binding) and

$$F_j \;=\; I_j(\lambda_k(X)\, a_j^{-1}) \;-\; w_{1j} \;\geq\; 0 \text{ for all } j \;\geq\; k \tag{42}$$

(constraint $F_j \geq 0$ is not binding for $j \geq k$) and (38) holds, that is,

$$a_j\, u'_j(w_{1j}) \;<\; \lambda_k(X) \tag{43}$$

for all $j < k$. First, let $k > J+1$. Recall now that

$$Z_{k+1} \;=\; \sum_{i=k+1}^{N} (I_i(a_i^{-1}\, a_k\, u'_k(w_{1\,k})) \;-\; w_{1i}) \;=\; \sum_{i=k}^{N} (I_i(a_i^{-1}\, a_k\, u'_k(w_{1\,k})) \;-\; w_{1i}),$$

and therefore $X \in [Z_{k+1}, Z_k]$ if and only if

$$\sum_{i=k}^{N} (I_i(a_i^{-1}\, a_k\, u'_k(w_{1\,k})) \;-\; w_{1i}) \;\leq\; X \;\leq\; \sum_{i=k}^{N} (I_i(a_i^{-1}\, a_{k-1}\, u'_{k-1}(w_{1\,k-1})) \;-\; w_{1i})\,.$$

Recalling that

$$X \;=\; \sum_{i\,=\,k}^{N} \left(I_i \left(a_i^{-1}\, \lambda_k(X)\right) \;-\; w_{1i}\right), \tag{44}$$

we get that

$$\lambda_k(X) \;\in\; [a_{k-1}\, u'_{k-1}(w_{1\,k-1})\,,\, a_k\, u'_k(w_{1\,k})]. \tag{45}$$

If $k = J+1$, the same argument implies that

$$\lambda_{J+1}(X) \;\in\; [u'_S(w_1)\,,\, a_{J+1}\, u'_{J+1}(w_{1\,J+1})]\,. \tag{46}$$

Recall that the investors are ordered in such a way that the sequence

$$Y_i \;=\; \frac{e^{-\rho_i}\, u_i'(w_{1i})}{u_i'(c_{0i})} \;=\; \frac{a_i\, u_i'(w_{1i})\, e^{-\rho_S}}{u_S'(c_{0S})}$$

is monotone decreasing in i, and the inequality

$$Y_i \;>\; Y_S$$

only holds true if $i \geq J+1$. Consequently,

$$a_N\, u_N'(w_{1N}) \;\geq\; \cdots \;\geq\; a_{J+1}\, u_{J+1}'(w_{1\,J+1}) \;\geq\; u_S'(w_1) \;\;\geq\;\; a_J\, u_J'(w_{1J}) \;\geq\; \cdots \;\geq\; a_1\, u_1'(w_{11})\,. \tag{47}$$

Inequalities (45), (46), and (47) immediately yield (41) and (43). Finally, for $j \geq k$,

$$a_j\, u_j'(w_{1j}) \;\;\geq\;\; a_k\, u_k'(w_{1k}) \;\;\Leftrightarrow\;\; a_j^{-1} \;\leq\; u_j'(w_{1j})\,(a_k\, u_k'(w_{1k}))^{-1}$$

and, using that $\lambda_k(X) \;\leq\; a_k\, u_k'(w_{1k})$, we get

$$F_j \;=\; I_j(\lambda_k(X)\, a_j^{-1}) \;-\; w_{1j} \;\;\geq\;\; I_j(a_k\, u_k'(w_{1k})\, u_j'(w_{1j})\,(a_k\, u_k'(w_{1k}))^{-1}) \;-\; w_{1j} \;\;=\;\; 0,$$

and (42) follows.

It remains to prove the identity for the derivative. Differentiating (39), we get

$$F_j'(X) \;\;=\;\; (u_j''(c_{1j}))^{-1}\, a_j^{-1}\, \lambda_k'(X),$$

and, differentiating (40), we get

$$\lambda_k'(X) \;\;=\;\; \frac{1}{\sum_{i\,\geq\,k} I_i'\, a_i^{-1}}\,.$$

Differentiating $u_i'(I_i'(z)) \;=\; z$ at $z = a_i^{-1}\lambda_k(X)$, we get

$$(u_i''(c_{1i}))^{-1} \;=\; I_i'(a_i^{-1}\,\lambda_k(X))\,.$$

Thus,

$$\begin{aligned} F_j'(X) \;&=\; \frac{(u_j''(c_{1j}))^{-1}\,a_j^{-1}}{\sum_{i\geq k} I_i'\,a_i^{-1}} \;=\; \frac{(u_j''(c_{1j}))^{-1}\,a_j^{-1}}{\sum_{i\geq k}(u_i''(c_{1i}))^{-1}\,a_i^{-1}} \\ &=\; \frac{(u_j''(c_{1j}))^{-1}\,\lambda_k(X)\,a_j^{-1}}{\sum_{i\geq k}(u_i''(c_{1i}))^{-1}\,\lambda_k(X)\,a_i^{-1}} \;=\; \frac{(u_j''(c_{1j}))^{-1}\,u_j'(c_{1j})}{\sum_{i\geq k}(u_i''(c_{1i}))^{-1}\,u_i'(c_{1i})}\,, \end{aligned} \tag{48}$$

which is what had to be proved. ∎

It remains to cover the case when the constraint $\sum_i F_i(X) \leq X$ is not binding. This is done in the following lemma.

Lemma B.2 *Let $k \leq J$. Then, for all $X \in [Z_{k+1}, Z_k]\,(= \text{Tranche}_k)$, the constraint $\sum_j F_j(X) \leq X$ is not binding, and the constraint $F_j(x) \geq 0$ is binding for all $j \leq k$. The optimal allocation for $X \in \text{Tranche}_k$ is uniquely determined via*

$$F_j(X) \;=\; \begin{cases} I_j(u_S'(w_1 + X - F(X))\,a_j^{-1}) \;-\; w_{1j}\,, & j \;>\; k \\ 0\,, & j \;\leq\; k\,. \end{cases} \tag{49}$$

Here, $F(X)$ is the unique solution to:

$$F(X) \;-\; \sum_{j\,\geq\,k+1} \big(I_j(u_S'(w_1 + X - F(X))\,a_j^{-1}) \;-\; w_{1j}\big) \;=\; 0. \tag{50}$$

The slope of $F_j(X),\; j \;\geq\; k+1$ satisfies

$$\frac{d}{dx}F_j(X) \;=\; \frac{R_j(c_{1j})}{R_S(c_{1S}) \;+\; \sum_{i>k} R_i(c_{1i})}\,.$$

Proof. We need to show that the allocation (49) and (50) satisfy the Kuhn-Tucker conditions:

$$a_i\, u_i'(w_{1i}\,+\,F_i)\;=\;u_S'(w_1+X-F(X)),$$

with $F_i \geq 0$ for all $i > k$ and

$$a_i\, u_i'(w_{1i})\;-u_S'(w_1+X-F(X))\;<\;0$$

for all $i\;\leq\;k$.

For simplicity let $k\;<\;J$. By assumption, $X \in [Z_{k+1}\,,\,Z_k]$; that is,

$$\begin{aligned} &I_S(a_k\, u_k'(w_{1k}))\;-\;w_1\;+\sum_{i\,:\,\geq k+1}\big(I_i\left(a_i^{-1}\, a_k\, u_k'(w_{1k})\right)\;-\;w_{1i}\big)\\ &>\;X\;>\;I_S(a_{k+1}\, u_{k+1}'(w_{1\,k+1}))\;-\;w_1\;+\sum_{i\,:\,\geq k+1}\big(I_i\left(a_i^{-1}\, a_{k+1}\, u_{k+1}'(w_{1k})\right)\;-\;w_{1i}\big)\,. \end{aligned} \tag{51}$$

We show that the unique solution F to (50) satisfies

$$w_1\;+\;X\;-\;I_S(a_k\, u_k'(w_{1\,k}))\;\leq\;F\;\leq\;w_1\;+\;X\;-\;I_S(a_{k+1}\, u_{k+1}'(w_{1\,k+1})). \tag{52}$$

Indeed,

$$\begin{aligned} &w_1\;+\;X\;-\;I_S(a_{k+1}\, u_{k+1}'(w_{1\,k+1}))\\ &\quad-\sum_{j\,\geq\, k+1}\big(I_j(u_S'(w_1+X-(w_1\;+\;X\;-\;I_S(a_{k+1}\, u_{k+1}'(w_{1\,k+1}))))\, a_j^{-1})\;-\;w_{1j}\big)\\ &\qquad\qquad=\;X\;-\;Z_{k+1}\;\geq\;0, \end{aligned} \tag{53}$$

and, similarly,

$$
\begin{aligned}
w_1 \,+\, X \,-\, I_S(a_k\, u_k'(w_{1\,k})) & \\
- \sum_{j \,\geq\, k+1} \big(I_j(u_S'(w_1 + X - (w_1 \,+\, X \,-\, I_S(a_k\, u_k'(w_{1\,k}))))\, a_j^{-1}) \,-\, w_{1j}\big) & \\
= X \,-\, Z_k \;\leq\; 0. & \quad (54)
\end{aligned}
$$

Consequently, by continuity and monotonicity, the right-hand side of (50) crosses zero at a single point F, satisfying (52). Hence, for $j \geq k+1$, by (47), we get:

$$
\begin{aligned}
F_j(X) \;=\; & I_j(u_S'(w_1 + X - F(X))\, a_j^{-1}) \,-\, w_{1j} \\
\geq\; & I_j(a_{k+1}\, u_{k+1}'(w_{1\,k+1})\, a_j^{-1}) \,-\, w_{1j} \;\geq\; I_j(a_j\, u_j'(w_{1\,j})\, a_j^{-1}) \,-\, w_{1j} \;=\; 0. \quad (55)
\end{aligned}
$$

It remains to be shown that the constraint $F_j(X) \geq 0$ is binding for $j \leq k$. By (52) and (47),

$$
a_j\, u_j'(w_{1j}) \,-\, u_S'(w_1 + X \,-F(X)) \;\leq\; a_j\, u_j'(w_{1j}) \,-\, a_k\, u_k'(w_{1k}) \;\leq\; 0,
$$

and the claim follows.

■

To complete the proof of Theorem 5.4, we only need to show that there is no trade if and only if (15) is violated. That is, the allocation $F_i = 0, i = 1, \cdots, N$ satisfies the first-order Kuhn-Tucker conditions if and only if (15) does not hold. Since in this case Y_i and Y_S coincide with the pre-trade MRIS, we need to show that

$$
\frac{e^{-\rho_i}\, u_i'(w_{1i})}{u_i'(w_{0i})} \;\leq\; \frac{e^{-\rho_S}\, u_S'(w_1 \,+\, X)}{u_S'(w_0)}
$$

for all $i \,=\, 1\,, \,\cdots\,, \,N$ and all $X \in [0, \bar{X}]$. Since $u_S'(c)$ is monotone decreasing in c, this holds if and only if

$$\max_i \frac{e^{-\rho_i}\, u_i'(w_{1i})}{u_i'(w_{0i})} \;\le\; \frac{e^{-\rho_S}\, u_S'(w_1 \,+\, \bar{X})}{u_S'(w_0)},$$

and the claim follows.

C Contraction Mapping

We prove here the following extended version of Lemma 6.1.

For each $i \,=\, 1\,,\, \cdots\,,\, N$, let:

$$\Omega^{-i} \;\stackrel{def}{=}\; \times_{j\neq i}\, [\beta_i^{\min}\,,\, \beta_i^{\max}]\,.$$

Lemma C.1 *Fix a constant* $C > 0$ *and let*

$$H_i(C, b_{-i})$$

be the unique solution to

$$H_i(C, b_{-i}) \,-\, e^{\rho_i}\, C\, u_i' \left(v_i \left(L_i - e^{-\rho_i} E[u_i(w_{1i} + F_i(X, (H_i(C, b_{-i}), b_{-i})))]\right)\right) \;=\; 0\,.$$

Then, H_i *is monotone increasing in* $C \in [C_{\min}\,,\, C_{\max}]$ *and* $b_{-i} \in e^{\Omega^{-i}}$ *and takes values in* $[e^{\beta_i^{\min}}\,,\, e^{\beta_i^{\max}}]$*. Furthermore, there exists an* $\eta < 1$ *such that*

$$\sum_{j\neq i} b_j \frac{\partial H_i}{\partial b_j} \;\le\; \eta\, H_i$$

for all $b_{-i} \in \Omega^{-i}$ *except for points in a finite union of hyperplanes, for which the derivatives do not exist.*

Proof of Lemma 6.1. Consider the function:

$$\psi_i(y\,,\,b_{-i}\,,\,C)\ \stackrel{def}{=}\ e^{\rho_i}\,C\,u_i'\left(v_i\left(L_i-e^{-\rho_i}E[u_i(w_{1i}+F_i(X,(y,b_{-i})))]\right)\right)\,.$$

Then, the defining equation for H_i can be rewritten as

$$H_i\ =\ \psi_i(H_i\,,\,b_{-i}\,,\,C)\,.$$

To complete the proof of the first part of the lemma, it remains to be shown that (1) ψ_i is monotone decreasing in y; (2) for each fixed $C\in[C_{\min}\,,\,C_{\max}]$ and each fixed b_{-i}, it maps the whole $\mathbb{R}$ into the compact interval $[e^{\beta_i^{\min}}\,,\,e^{\beta_i^{\max}}]$; and (3) it is monotone increasing in $b_{-i}\,,\,C$ and is piecewise C^1 with respect to all variables.

By definition, the form of the function F_i depends on the relative ranking of investors, which, in turn, is determined by the ordering of the numbers $b_i/u_i'(w_{1i})$ (see (47)). For each permutation π of $\{1,\cdots,N\}$, define the corresponding "sector": the subset of $\mathbb{R}_+^n$ such that, for all $\mathbf{b}$ in this sector, the sequence $b_{\pi(i)}/u_{\pi(i)}'(w_{1\pi(i)})$ is monotone increasing in i. The borders of these sectors belong to hyperplanes for which $b_iu_i'(w_{1i})=b_ju_j'(w_{1j})$ for some $i\neq j$.

Clearly, since the function ψ_i is continuous, it suffices to prove the required result for each fixed sector.[34]

As above, by abuse of notation, we reorder the investors for each fixed sector so that (47) holds, and thus investor i will mean the investor whose rank is equal to i.

First, the fact that the image of the function ψ_i is inside the interval $[e^{\beta_i^{\min}}\,,\,e^{\beta_i^{\max}}]$

[34]Here, one should in general take additional care of the situation when H_i hits the boundaries of the sectors for an open set of parameters. Clearly, this cannot happen for generic values of parameters (discount rates and endowments), and we therefore ignore it. The proof can be easily modified to cover this non-generic situation.

follows directly from the definition and the inequality:

$$0 \ \le \ F_i(X) \ \le \ X.$$

Now, we need the following auxiliary.

Lemma C.2 *For any X inside a tranche, F_i is a piecewise C^1-function of $\mathbf{b}$ and satisfies*

$$\frac{\partial F_i}{\partial b_i} \ \le \ 0$$

and

$$\frac{\partial F_i}{\partial b_j} \ \ge 0$$

for all $j \neq i$. Furthermore,

$$-b_i \, \frac{\partial F_i}{\partial b_i} \ \ge \ \sum_{j \neq i} b_j \, \frac{\partial F_i}{\partial b_j}\,.$$

Proof. Suppose first that we are in the regime $F(X) < X$. Then, by (49),

$$F_i(X) \ = \ I_i(b_i \, u_S'(w_1 + X - F(X))) - w_{1i},$$

and

$$F(X) \ = \ F(\mathbf{b}\,,\, x)$$

solves

$$F(X) \ - \ \sum_j I_j(b_j \, u_S'(w_1 + X - F(X))) \ = \ 0\,.$$

Here, the summation is only over those investors j that participate in the tranche. Thus,

$$\frac{\partial F}{\partial b_j} \ = \ \frac{I_j'(b_j \, u_S'(c_{1S}))\, u_S'(c_{1S})}{1 \, + \, \sum_k \, I_k'(b_k \, u_S'(c_{1S}))\, b_k \, u_S''(c_{1S})};$$

and, hence, for $j \neq i$,

$$\frac{\partial F_i}{\partial b_j} \;=\; -I_i'(b_i\,u_S'(c_{1S}))\,b_i\,u_S''(c_{1S})\,\frac{I_j'(b_j\,u_S'(c_{1S}))\,u_S'(c_{1S})}{1\,+\,\sum_k\,I_k'(b_k\,u_S'(c_{1S}))\,b_k\,u_S''(c_{1S})} \;>\; 0$$

if investor j participates in the tranche, and the derivative is zero otherwise. Consequently,

$$\sum_{j\neq i} b_j \frac{\partial F_i}{\partial b_j} \;=\; -(I_i'(b_i\,u_S'(c_{1S})))\,b_i\,u_S'(c_{1S})\,\frac{\sum_{k\neq i}\,I_k'(b_k\,u_S'(c_{1S}))\,b_k\;u_S''(c_{1S})}{1\,+\,\sum_k\,I_k'(b_k\,u_S'(c_{1S}))\,b_k\,u_S''(c_{1S})}$$

and

$$\begin{aligned}\frac{\partial F_i}{\partial b_i} \;&=\; I_i'(b_i\,u_S'(c_{1S}))u_S'(c_{1S}) \\ &-\; I_i'(b_i\,u_S'(c_{1S}))\,b_i\,u_S''(c_{1S})\,\frac{I_i'(b_i\,u_S'(c_{1S}))\,u_S'(c_{1S})}{1\,+\,\sum_k\,I_k'(b_k\,u_S'(c_{1S}))\,b_k\,u_S''(c_{1S})} \\ &=\; I_i'(b_i\,u_S'(c_{1S}))u_S'(c_{1S})\,\left(1\;-\;\frac{I_i'(b_i\,u_S'(c_{1S}))\,b_i\,u_S''(c_{1S})}{1\,+\,\sum_k\,I_k'(b_k\,u_S'(c_{1S}))\,b_k\,u_S''(c_{1S})}\right) \\ &=\; I_i'(b_i\,u_S'(c_{1S}))u_S'(c_{1S})\,\frac{1\,+\,\sum_{k\neq i}\,I_k'(b_k\,u_S'(c_{1S}))\,b_k\,u_S''(c_{1S})}{1\,+\,\sum_k\,I_k'(b_k\,u_S'(c_{1S}))\,b_k\,u_S''(c_{1S})}\,.\end{aligned}$$

Therefore,

$$-b_i\,\frac{\partial F_i}{\partial b_i} \;>\; \sum_{j\neq i} b_j\,\frac{\partial F_i}{\partial b_j}\,.$$

Suppose now that the constraint $F(X) \leq X$ is binding, so that $F(X) = X$. Then, by (39),

$$F_i(X) \;=\; I_i(\lambda(X)\,b_i)\;-\,w_{1i},$$

with $\lambda(X)$ solving

$$X \;-\; \sum_i (I_i(\lambda(X)\,b_i)\;-\,w_{1i}) \;=\; 0$$

where the summation is only over investors i participating in the tranche. Differentiating, we get

$$\frac{\partial \lambda(X)}{\partial b_j} \;=\; \frac{-I_j'(b_j\,\lambda(X))\,\lambda(X)}{\sum_k\, I_k'(b_k\,\lambda(X))\,b_k}$$

and, hence,

$$\frac{\partial F_i}{\partial b_j} \;=\; -\,b_i\,I_i'(\lambda(X)\,b_i)\,\frac{I_j'(b_j\,\lambda(X))\,\lambda(X)}{\sum_k\, I_k'(b_k\,\lambda(X))\,b_k} \;>\; 0$$

if the investor $j \neq i$ participates in the tranche and the derivative is zero otherwise. Similarly,

$$\begin{aligned}\frac{\partial F_i}{\partial b_i} \;&=\; I_i'(\lambda(X)\,b_i)\,\lambda(X) \;-\; b_i\,I_i'(\lambda(X)\,b_i)\,\frac{I_i'(b_i\,\lambda(X))\,\lambda(X)}{\sum_k\, I_k'(b_k\,\lambda(X))\,b_k}\\ &=\; I_i'(\lambda(X)\,b_i)\,\lambda(X)\,\frac{\sum_{k\neq i}\, I_k'(b_k\,\lambda(X))\,b_k}{\sum_k\, I_k'(b_k\,\lambda(X))\,b_k} \;<\; 0\,. \qquad (56)\end{aligned}$$

if $F_i(X) \neq 0$ (that is, if investor i participates in the tranche), and is zero otherwise. A direct calculation implies that

$$-b_i\,\frac{\partial F_i}{\partial b_i} \;=\; \sum_{j\neq i} b_j\,\frac{\partial F_i}{\partial b_j}\,.$$

■

Note that the function $F_i(x)$ is continuous and is a smooth function of all b_i as long as $\mathbf{b}$ varies inside a fixed sector. Therefore,

$$\begin{aligned}&\frac{\partial}{\partial b_k}\,E[u_i(w_{1i}+F_i(\mathbf{b},X))]\\ &=\; \frac{\partial}{\partial b_k}\sum_j \int_{Z_{j+1}}^{Z_j} u_i(w_{1i}+F_i(\mathbf{b},x))p(x)dx\\ &=\; \sum_j \int_{Z_{j+1}}^{Z_j} u_i'(w_{1i}+F_i(\mathbf{b},x))\left(\frac{\partial}{\partial b_k}F_i(\mathbf{b},x)\right)p(x)dx\\ &=\; E\left[u_i'(w_{1i}+F_i(\mathbf{b},X))\left(\frac{\partial}{\partial b_k}F_i(\mathbf{b},X)\right)\right]\,.\end{aligned} \qquad (57)$$

The derivatives of $Z_i(b_j)$ do not appear on the right-hand side of (57) because the boundary terms cancel, as in (29).

Denote

$$\tilde{c}_{i0} \;=\; v_i\left(L_i - e^{-\rho_i} E[u_i(w_{1i} + F_i(X, (H_i(C, b_{-i}), b_{-i})))]\right).$$

Then, using the identity $v_i'(x) \;=\; (u_i'(v(x)))^{-1}$, we get

$$\frac{\partial H_i}{\partial b_j} \;=\; \frac{C\, A_i(\tilde{c}_{i0})\, E\left[u_i'(w_{1i} + F_i(X))\left(\frac{\partial}{\partial b_j} F_i(X)\right)\right]}{1 \,-\, C\, A_i(\tilde{c}_{i0})\, E\left[u_i'(w_{1i} + F_i(X))\left(\frac{\partial}{\partial b_i} F_i(X)\right)\right]}$$

where

$$A_i(x) \;=\; -\frac{u_i''(x)}{u_i'(x)}.$$

Lemma C.2 implies that

$$\begin{aligned} \sum_{j \neq i} b_j \frac{\partial H_i}{\partial b_j} \;&=\; \frac{C\, A_i(\tilde{c}_{i0})\, E\left[u_i'(w_{1i} + F_i(X)) \sum_j b_j \left(\frac{\partial}{\partial b_j} F_i(X)\right)\right]}{1 \,-\, C\, A_i(\tilde{c}_{i0})\, E\left[u_i'(w_{1i} + F_i(X))\left(\frac{\partial}{\partial b_i} F_i(X)\right)\right]} \\ &\leq\; \frac{-C\, A_i(\tilde{c}_{i0})\, E\left[u_i'(w_{1i} + F_i(X))\, b_i \left(\frac{\partial}{\partial b_i} F_i(X)\right)\right]}{1 \,-\, C\, A_i(\tilde{c}_{i0})\, E\left[u_i'(w_{1i} + F_i(X))\left(\frac{\partial}{\partial b_i} F_i(X)\right)\right]} \;\leq\; \eta\, b_i \;=\; \eta\, H_i \end{aligned} \tag{58}$$

where we have defined

$$\eta \;=\; \max_{e^{\Omega}} \frac{-C\, A_i(\tilde{c}_{i0})\, E\left[u_i'(w_{1i} + F_i(X))\left(\frac{\partial}{\partial b_i} F_i(X)\right)\right]}{1 \,-\, C\, A_i(\tilde{c}_{i0})\, E\left[u_i'(w_{1i} + F_i(X))\left(\frac{\partial}{\partial b_i} F_i(X)\right)\right]}.$$

It follows from the proof of Lemma C.2 that the derivative $\frac{\partial}{\partial b_i} F_i(X)$ stays uniformly bounded when $\mathbf{b}$ varies on the compact subset e^{Ω} and therefore $\eta < 1$. The proof is complete. ■

Lemma C.3 *Consider a map $G = (G_i) : \Omega \to \Omega$ with coordinate maps $G_i(b_1, \cdots, b_N)$, such that the following is true:*

- *The map G is continuous;*
- *There exists a finite set $\mathcal{S}$ of smooth hyper-surfaces such that G is C^1 on $\Omega \setminus \mathcal{S}$; and*
- *There exists a constant $\eta < 1$ such that*

$$\sum_j \left| \frac{\partial G_i}{\partial d_j} \right| \;\leq\; \eta$$

for all i and all $\mathbf{d} = (d_j) \in \Omega \setminus \mathcal{S}$.

Then, the map G is a contraction in the l_∞ norm $\|\mathbf{d}\|_{l_\infty} \;=\; \max_i |d_i|$, so that

$$\|G(\mathbf{d}^1) - G(\mathbf{d}^2)\|_{l_\infty} \;\leq\; \alpha \, \|\mathbf{d}^1 - \mathbf{d}^2\|_{l_\infty}.$$

In particular, G has a unique fixed point $\mathbf{d}_*$ that satisfies

$$\mathbf{d}^* \;=\; \lim_{n\to\infty} G^n(\mathbf{d}^0)$$

for any $\mathbf{d}^0 \in \Omega$.

Proof of Lemma C.3. With continuity, we may assume that the two points $\mathbf{d}^1$ and $\mathbf{d}^2$ are in a generic position, so that the segment,

$$\mathbf{d}(t) \;=\; \mathbf{d}^1 \,+\, t\,(\mathbf{d}^2 - \mathbf{d}^1)\;,\; t \in [0,1]$$

connecting $\mathbf{d}^1$ and $\mathbf{d}^2$, intersects the hyperplanes from $\mathcal{S}$ for a finite set

$$t_1 \;<\; t_2 \;<\; \cdots \;<\; t_{m+1}.$$

Then,

$$
\begin{aligned}
|G_i(\mathbf{d}^1) - G_i(\mathbf{d}^2)| \;&=\; \left| \sum_{k=1}^{m} \int_{t_k}^{t_{k+1}} \sum_j \frac{\partial G_i}{\partial d_j}(\mathbf{d}(t))\,(d_j^2 - d_j^1)\,dt \right| \\
&\le\; \max_j |d_j^2 - d_j^1|\,\eta \;=\; \eta\,\|\mathbf{d}^1 - \mathbf{d}^2\|_{l_\infty}.
\end{aligned}
\tag{59}
$$

The last claim follows from the contraction mapping theorem (see Lucas and Stockey (1989), Theorem 3.2 on p. 50). ■

Proof of Lemma 6.2. Let $b_j = e^{d_j}$. By Lemma C.1,

$$
\sum_j \frac{\partial (G_C)_i}{\partial d_j} \;=\; \sum_{j\neq i} (H_i)^{-1} \frac{\partial H_i}{\partial b_j} b_j \;\le\; \alpha,
$$

and the claim follows from Lemma C.3. ■

Proof of Lemma 6.5. Pick a parameter ζ and suppose that

$$
G_C(\mathbf{d}, \zeta_1) \;\ge\; G_C(\mathbf{d}, \zeta_2)
$$

for all $\mathbf{d}$ and, for any fixed $\mathbf{d} = (d_i)$, the expression

$$
\left((1-\alpha)\, e^{\rho_S}\, u_S' \left(w_0 \,+\, (1-\alpha) \sum_i \left(w_{0i} \,-\, I_i(e^{d_i}\, e^{-\rho_i}\, C^{-1}) \right) \right) \right)^{-1}
$$

is larger for ζ_1 than for ζ_2. Pick a point $\mathbf{d}_0 \in \Omega$. Then, since G_C is monotone increasing in $\mathbf{d}$, we get:

$$
\begin{aligned}
G_C^2(\mathbf{d}_0, \zeta_1) \;&=\; G_C(G_C(\mathbf{d}_0, \zeta_1), \zeta_1) \;\ge\; G_C(G_C(\mathbf{d}_0, \zeta_2), \zeta_1) \\
&\ge\; G_C(G_C(\mathbf{d}_0, \zeta_2), \zeta_2) \;=\; G_C^2(\mathbf{d}_0, \zeta_2).
\end{aligned}
\tag{60}
$$

Repeating the same argument, we get:

$$G_C^n(\mathbf{d}_0, \zeta_1) \;\ge\; G_C^n(\mathbf{d}_0, \zeta_2)$$

for any $n \in \mathbb{N}$. Sending $n \to \infty$ and using Lemma 6.2 and Lemma C.3, we get:

$$\mathbf{d}^*(C, \zeta_1) \;\ge\; \mathbf{d}^*(C, \zeta_2)$$

for any C. This immediately yields that $C^*(\zeta_1) \ge C^*(\zeta_2)$, and therefore

$$\mathbf{d}^*(C^*(\zeta_1)\,,\,\zeta_1) \;\ge\; \mathbf{d}^*(C^*(\zeta_2)\,,\,\zeta_1) \;\ge\; \mathbf{d}^*(C^*(\zeta_2)\,,\,\zeta_2)$$

and the claim follows. ■

Lemma C.4 *Suppose that an increase in a parameter ζ leads to a decrease in the optimal $\mathbf{d}^*$. Then, this also leads to more selling.*

Proof of Lemma C.4. A decrease in $d_i\,,\ i = 1\,,\,\cdots\,,\,N$ is equivalent to an increase in all coordinates of $\mathbf{a} = (a_i) \;=\; (e^{-d_i})$. Consequently, the number of the coordinates of $\mathbf{a}$ for which $a_i\,u_i'(w_{1i}) > u_S'(w_1)$ increases. This is precisely $\#\{\text{senior}\}$. Similarly, by definition, $Z_{\text{full selling}} = Z_{J+1}$ is monotone increasing in all a_i (see (13)), and $Z_{\text{no trade}}$ is monotone decreasing in all coordinates of $\mathbf{a}$. Finally, the participation index is equal to 1 if $a_i\,u_i'(w_{1i}) > u_S'(w_1)$ and therefore stays equal to 1 if a_i increases. ■

Proof of Proposition 6.7. By the definition of FOSD dominance, an increase in the distribution of X in the FOSD sense leads to an increase of

$$E[u_i(w_{1i} \,+\, F_i(\mathbf{b}, X))]\,,$$

for all $i = 1\,,\,\cdots\,,\,N$ and, consequently, to an increase in the right-hand side of (18) for any fixed **a**. Therefore, the solution H_i to (18) also increases in response to this change in the distribution of X. By Lemma 6.5, this leads to an increase of all coordinates of vector **b**. The claims follow now from Lemma C.4.

Similarly, an increase in w_0 and a decrease in ρ_S lead to an increase in the right-hand side of (19). This leads to an increase in C, and therefore, by Lemma 6.5, all coordinates of vector **b** increase. ■

Proof of Proposition 7.3. A direct calculation shows that, under the CARA assumption, the vector $\mathbf{b} = (b_i)$ solves

$$b_i \;=\; e^{\rho_i}\, C\, \left(e^{-A_i\, w_{0i}} \;+\; e^{-\rho_i - A_i w_{1i}}\, E[1 - e^{-A_i\, F_i(X)}]\right)\ ,\ i \;=\; 1\,,\,\cdots\,,\,N. \tag{61}$$

Suppose that

$$\frac{e^{-\rho_i}\, e^{-A_i\, w_{1i}}}{e^{-A_i\, w_{0i}}} \;>\; \frac{e^{-\rho_j}\, e^{-A_j\, w_{1j}}}{e^{-A_j\, w_{0j}}} \tag{62}$$

for some investors i and j, but $\mathrm{rank}(i) \,<\, \mathrm{rank}(j)$. By definition, this means that

$$b_i\, e^{A_i\, w_{1i}} \;\geq\; b_j\, e^{A_j\, w_{1j}}. \tag{63}$$

We now claim that the inequality $\mathrm{rank}(i) \,<\, \mathrm{rank}(j)$ implies

$$A_i\, F_i \;\leq\; A_j\, F_j. \tag{64}$$

Indeed, for all tranches in which investor i participates, the slopes of $A_i\, F_i$ and $A_j\, F_j$ coincide by Proposition 5.5. Since j has a higher rank, $A_i\, F_i(X) \,=\, 0$ for all X for which $A_j\, F_j(X) = 0$. The claim (64) follows now by continuity of F_i and F_j. Consequently,

$$E[1 - e^{-A_i\, F_i(X)}] \;\leq\; E[1 - e^{-A_j\, F_j(X)}],$$

and therefore (61) and (62) together yield

$$
\begin{aligned}
b_i\, e^{A_i\, w_{1i}} \;&=\; e^{\rho_i}\, C\, e^{A_i\,(w_{1i}-w_{0i})} \,+\, C\, E[1-e^{-A_i\, F_i(X)}] \\
&<\; e^{\rho_j}\, C\, e^{A_j\,(w_{1j}-w_{0j})} \,+\, C\, E[1-e^{-A_j\, F_j(X)}] \;=\; b_j\, e^{A_j\, w_{1j}},
\end{aligned} \tag{65}
$$

which contradicts (63). The proof is complete. ■

Proof of Proposition 8.2. Since the issuer is risk neutral, $C = e^{-\rho_S}$. An increase in w_{0i} leads to an decrease in all coordinates of the map G_C, and the claim follows from Lemmas 6.5 and C.4.

Now, consider the optimal allocation with $N+1$ investors and let $(\tilde{\mathbf{d}}\,,\, d_{N+1})$ be the corresponding vector, with the coordinate d_{N+1} corresponding to the new, $(N+1)$th, investor. Let $\tilde{G}_C$ be the map of Lemma 6.2, corresponding to the case of $N+1$ investors. Further, let $\tilde{G}_C^{(N)}$ be the "submap" of $\tilde{G}_C$, consisting of the first N coordinates of $\tilde{G}_C$. Finally, let G_C be the map corresponding to the N investor case. Then, the vector $\tilde{\mathbf{d}}$ satisfies

$$
\tilde{\mathbf{d}} \;=\; \tilde{G}_C^{(N)}(\tilde{\mathbf{d}}, d_{N+1}).
$$

Similarly, the vector $\mathbf{d}$ corresponding to the N investor case solves

$$
\mathbf{d} \;=\; G_C(\mathbf{d}).
$$

Now, let $\tilde{d}_{N+1} \;=\; \max\{d_{N+1}\,,\, \log(u'_{N+1}(w_{1\,N+1})\}$. By definition, $\tilde{d}_{N+1} \;\ge\; d_{N+1}$. Furthermore, such a large $\tilde{d}_{N+1}$ will correspond to a small $\tilde{a}_{N+1}$, satisfying

$$
\tilde{a}_{N+1}\, u'_{N+1}(w_{1\,N+1}) \;\le\; 1 \;=\; u'_S(w_1).
$$

Therefore, by (47), such a $\tilde{d}_{N+1}$ will not change the tranche structure[35] and, consequently,

$$\tilde{G}_C^{(N)}(\mathbf{x}\,,\,\tilde{d}_{N+1}) \;=\; G_C(\mathbf{x})$$

for any $\mathbf{x}$. Thus, we have:

$$\tilde{\mathbf{d}} \;=\; \tilde{G}_C^{(N)}(\tilde{\mathbf{d}}, d_{N+1}) \;\leq\; \tilde{G}_C^{(N)}(\tilde{\mathbf{d}}, \tilde{d}_{N+1}) \;=\; G_C(\tilde{\mathbf{d}})\,.$$

Applying G_C to this inequality repeatedly and using monotonicity of G_C, we get:

$$\tilde{\mathbf{d}} \;\leq\; G_C(\tilde{\mathbf{d}}) \;\leq\; G_C(G_C(\tilde{\mathbf{d}})) \;\leq\; \cdots \;\leq\; (G_C)^n(\tilde{\mathbf{d}}).$$

Sending $n \to \infty$, we get, by Lemma 6.2, that

$$\tilde{\mathbf{d}} \;\leq\; \mathbf{d}.$$

The required result follows now from Lemma C.4. ■

Proof of Proposition 8.4. Let $\delta = e^{\rho_i}$. Consider the function:

$$f(\delta) \;=\; \delta\, u_i'\Big(v_i\big(u_i(w_{0i}) - \delta^{-1}\, E[u_i(w_{1i} + \phi(X)) \;-\; u_i(w_{1i})]\big)\Big)$$

for any fixed $0 \leq \phi(X) \leq X$. Then,

$$\begin{aligned}\frac{\partial}{\partial \delta}\, f(\delta) \;=\;& u_i'\Big(v_i\big(u_i(w_{0i}) - \delta^{-1}\, E[u_i(w_{1i} + \phi(X)) \;-\; u_i(w_{1i})]\big)\Big)\\ +\; \delta^{-1}\, E[u_i(w_{1i} + \phi(X)) \;-\; u_i(w_{1i})]\, & u_i''\Big(v_i\big(u_i(w_{0i}) - \delta^{-1}\, E[u_i(w_{1i} + \phi(X) \;-\; u_i(w_{1i})]\big)\Big)\\ & \times\, v_i'\big(u_i(w_{0i}) - \delta^{-1}\, E[u_i(w_{1i} + \phi(X)) \;-\; u_i(w_{1i})]\big)\,. \quad (66)\end{aligned}$$

[35] Since the issuer is risk neutral, tranches with indices smaller than $J+1$ are not sold.

Denote

$$z \;=\; v_i\big(u_i(w_{0i}) - \delta^{-1}\, E[u_i(w_{1i} + \phi(X)) \;-\; u_i(w_{1i})]\big).$$

Then,

$$w_{0i} \;\geq z \;\geq\; v_i\big(u_i(w_{0i}) - \delta^{-1}\, E[u_i(w_{1i} + X) \;-\; u_i(w_{1i})]\big) \;=\; w_{0i} \;-\; P_i^{\max}\,.$$

Using the identity $v_i'(x) \;=\; (u_i'(v_i(x)))^{-1}$, we get:

$$\begin{aligned} \frac{\partial}{\partial\delta}\, f(\delta) \;&=\; u_i'(z) \;+\; \delta^{-1} E[u_i(w_{1i} + \phi(X)) \;-\; u_i(w_{1i})]\, \frac{u_i''(z)}{u_i'(z)} \\ &=\; u_i'(z) \;-\; (u_i(w_{0i}) - u_i(z))\, \frac{1}{R_i(z)}\,. \end{aligned} \tag{67}$$

By assumption, the right-hand side is nonnegative. Hence, $f(\delta)$ is monotone increasing. Therefore, by Lemma 6.5, the coordinates of the vector $\mathbf{d}^*(C)$ are increasing in ρ_i, and the claim follows since the risk neutrality of the issuer implies $C = e^{-\rho_S}$. ■

References

Alvarez, Fernando and Urban J. Jermann (2000): "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica,* **68(4)**, 775-797.

Alvarez, Fernando and Urban J. Jermann (2001): "Quantitative Asset Pricing Implications of Endogenous Solvency Constraints," *Review of Financial Studies,* **14(4)**, 1117-1151.

Acharya, Viral and Alberto Bisin (2005): "Optimal Financial-Market Integration and Security Design," *Journal of Business,* **78(6)**, 2397-2433.

Allen, Franklin and Douglas Gale (1988): "Optimal Security Design," *Review of Financial Studies,* **1(3)**, 229-263.

Allen, Franklin and Douglas Gale (1991): "Arbitrage, Short Sales, and Financial Innovation," *Econometrica,* **59(4)**, 1041-1068.

Allen, Franklin and Douglas Gale (1994): *Financial Innovation and Risk Sharing,* Cambridge, MA: MIT Press.

Allen, Franklin, and Andrew Winton (1995): "Corporate Financial Structure, Incentives, and Optimal Contracting," in *Handbooks in Operations Research and Management Science,* vol. 9: Finance (Elsevier Science B.V.), ed. by R. A. Jarrow, V. Maksimovic and W. T. Ziemba.

An, Xudong, Yongheng Deng, and Anthony B. Sanders (2008): "Subordination Levels in Structured Financing," in *Corporate Finance, Volume 3: Financial Intermediation and Banking* (Elsevier Science), ed. by Arnoud W. A. Boot and Anjan V. Thakor.

Axelson, Ulf (2007): "Security Design with Investor Private Information," *Journal of Finance,* **62(6)**, 2587-2632.

Bolton, Patrick and David S. Scharfstein (1996): "Optimal Debt Structure and the Number of Creditors," *Journal of Political Economy,* **104(1)**, 1-25.

Boot, Arnoud W. A. and Anjan V. Thakor (1993): "Security Design," *Journal of Finance,* **48(4)**, 1349-1378.

Borch, Karl (1962): "Equilibrium in a Reinsurance Market," *Econometrica,* **30(3)**, 424-444.

Crémer, Jacques, and Richard P. McLean (1985): "Optimal Selling Strategies Under Uncertainty for a Discriminating Monopolist When Demands Are Interdependent," *Econometrica,* **53(2)**, 345-361.

DeMarzo, Peter and Darrell Duffie (1999): "A Liquidity-Based Model of Security Design," *Econometrica,* **67(1)**, 65-99.

DeMarzo, Peter (2005): "The Pooling and Tranching of Securities: A Model of Informed Intermediation," *Review of Financial Studies,* **18(1)**, 1-35.

DeMarzo, Peter, Ilan Kremer, and Andrzej Skrzypacz (2005): "Bidding with Securities: Auctions and Security Design," *American Economic Review,* **95(4)**, 936-959.

Diamond, Douglas (1993): "Seniority and Maturity of Debt Contracts," *Journal of Financial Economics,* **33**, 341-368.

Duffie, Darrell and Nicolae Garleanu (2001): "Risk and Valuation of Collateralized Debt Obligations," *Financial Analysts Journal,* Jan.-Feb. 2001, 41-59.

Duffie, Darrell and Rohit Rahi (1995): "Financial Market Innovation and Security Design: an Introduction," *Journal of Economic Theory,* **65(1)**, 1-42.

Fabozzi, Frank J., Henry Davis, and Moorad Choudhry (2006): *Introduction to Structured Finance.* Hoboken, NJ: John Wiley & Sons.

Fender, Ingo and Janet Mitchell (2005): "Structured Finance: Complexity, Risk, and the Use of Ratings," *Bank of International Settlements Quarterly Review,* June 2005, 67-79.

Franke, Günter, Markus Herrmann, and Thomas Weber (2007): "Information Asymmetries and Securitization Design," *working paper.*

Franke, Günter and Thomas Weber (2009): "Optimal Tranching in CDO-Transactions," *working paper.*

Fulghieri, Paolo and Dmitry Lukin (2001): "Information Production, Dilution Costs, and Optimal Security Design," *Journal of Financial Economics*, **61(1)**, 3-41.

Gollier, Christian (2004): *The Economics of Risk and Time.* Cambridge, MA: MIT Press.

Gorton, Gary B. and George G. Pennacchi (1990): "Financial Intermediaries and Liquidity Creation," *Journal of Finance,* **45(1)**, 49-71.

Harris, Milton and Artur Raviv (1991): "The Theory of Capital Structure," *Journal of Finance,* **46(1)**, 297-355.

Harris, Milton and Artur Raviv (1992): "The Theory of Security Design: A Survey," in *Advances in Economic Theory*, ed. by J.-J. Laffont, Cambridge: Cambridge University Press.

Hartman-Glaser, Barney, Tomasz Piskorski, and Alexei Tchistyi (2009): "Optimal Securitization with Moral Hazard," *working paper.*

Lucas, Robert E. and Nancy L. Stokey (1989): *Recursive Methods in Economic Dynamics*, Boston, MA: Harvard University Press.

Madan, Dilip and Badih Soubra (1991): "Design and Marketing of Financial Products," *Review of Financial Studies,* **4**, 361-384.

Mitter, Sanjoy K. (2008): "Convex Optimization in Infinite Dimensional Spaces," *in Recent Advances in Learning and Control, LNCIS 371, V.D. Blondel et al. (Eds.)*, 161-179.

Modigliani, Franco and Merton Miller (1958): "The Cost of Capital, Corporation Finance, and the Theory of Investment," *American Economic Review,* **48(3)**, 261-297.

Mitchell, Janet (2004): "Financial Intermediation Theory and the Sources of Value in Structured Finance Markets," *working paper.*

Piskorski, Tomasz and Alexei Tchistyi (2009): "Optimal Mortgage Design," *working paper.*

Raviv, Artur (1979): "The Design of an Optimal Insurance Policy," *American Economic Review,* **69(1)**, 84-96.

Ross, Stephen A (1989): "Institutional Markets, Financial Marketing, and Financial Innovation," *Journal of Finance,* **44**, 541-556.

Sharpe, William F. (1964): "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance,* **19(3)**, 425-442.

Seierstad, Atle and Knut Sydsaeter (1977): "Sufficient Conditions in Optimal Control Theory," *International Economic Review,* **18(2)**, 367-391.

Stiglitz, Joseph E. (1974): "On the Irrelevance of Corporate Financial Policy," *American Economic Review,* **64(6)**, 851-866.

Tchistyi, Alexei (2009): Security Design with Correlated Hidden Cash Flows: The Optimality of Performance Pricing," *working paper.*

Wilson, Robert (1968): "The Theory of Syndicates," *Econometrica*, **36 (1)**, 119-132.

Winton, Andrew (1995): "Costly State Verification and Multiple Investors: The Role of Seniority," *Review of Financial Studies,* **8(1)**, 91-123.